

What is a proof? It is a sequence of deductions from assumptions and a small number of elementary rules, leading to a conclusion. We will describe these rules in detail and explain how to use them in practice.

1. Logical consequence

1.1. Assumptions & conclusion

We have formulas $\mathcal{P}_1, \dots, \mathcal{P}_l$, which will be our **assumptions** (or **premises**), and a formula $\mathcal{Q}$, which will be our **conclusion**. Our aim is, of course, to move from true assumptions to a true conclusion. There will be two ways of doing this, and this is truly at the heart of Gödel's theory.

Remark. If there are finitely many assumptions $\mathcal{P}_1, \dots, \mathcal{P}_l$, we could always reduce them to the single assumption $\mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \dots \wedge \mathcal{P}_l$; in practice, we will not do this. We might also want several formulas as conclusions, but we will not do that either, since it would amount to studying them one by one.

1.2. Logical consequence $\models$

Let us begin with the notion of *logical consequence*, also called *semantic consequence*.

Definition.

We say that $\mathcal{Q}$ is a **logical consequence** of $\mathcal{P}_1, \dots, \mathcal{P}_l$ if every valuation that makes the formulas $\mathcal{P}_1, \dots, \mathcal{P}_l$ true also makes the formula $\mathcal{Q}$ true. We write:

$$\boxed{\mathcal{P}_1, \dots, \mathcal{P}_l \models \mathcal{Q}}$$

The symbol $\models$ is the *double turnstile*; it is read “entails.” In other words, if the assumptions hold, then the conclusion is true. (If one of the assumptions $\mathcal{P}_i$ is false, then nothing is required of $\mathcal{Q}$, which may be true or false.)

1.3. Truth table

Establishing a logical consequence

$$\mathcal{P}_1, \dots, \mathcal{P}_l \models \mathcal{Q}$$

is a matter of checking. We construct the truth table for all the propositional variables $P_1, \dots, P_n$ occurring in the formulas $\mathcal{P}_1, \dots, \mathcal{P}_l$ and $\mathcal{Q}$. In every row in which all the formulas $\mathcal{P}_1, \dots, \mathcal{P}_l$ evaluate to True, $\mathcal{Q}$ must also evaluate to True.

1.4. Examples

Example.

$$P \vee Q, \neg Q \models P$$

Here we have:

- two propositional variables P and Q ,
- two assumptions: $\mathcal{P}_1$: “ $P \vee Q$ ” and $\mathcal{P}_2$: “ $\neg Q$ ”,
- one conclusion: “ P .”

Here is the truth table:

P	Q	$P \vee Q$	$\neg Q$	P
T	T	T	F	—
T	F	T	T	T
F	T	T	F	—
F	F	F	T	—

Only the second row makes both assumptions true, and we then check that the conclusion is also true. We do not need to know whether the conclusion is true or false in the other cases. Thus we do indeed have the logical consequence $P \vee Q, \neg Q \models P$.

Let us emphasize that when we write

$$\mathcal{P}_1, \dots, \mathcal{P}_l \models \mathcal{Q}$$

we do not require the formulas $\mathcal{P}_1, \dots, \mathcal{P}_l$ always to evaluate to True. What we require is that *when* they evaluate to True, so does $\mathcal{Q}$. An extreme situation is, for example,

$$P, \neg P \models \text{anything}$$

This logical consequence is valid, whatever the conclusion, because P and $\neg P$ never evaluate to true at the same time.

Example.

$$P \rightarrow Q, Q \rightarrow R \models P \rightarrow R$$

Here we have three propositional variables P, Q, R , two assumptions, and one conclusion.

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$P \rightarrow R$
T	T	T	T	T	T
T	T	F	T	F	—
T	F	T	F	T	—
T	F	F	F	T	—
F	T	T	T	T	T
F	T	F	T	F	—
F	F	T	T	T	T
F	F	F	T	T	T

We consider only the rows in which both assumptions hold, and then check that the conclusion is also

true. We do not need to know whether the conclusion is true or false in the other cases. Thus we do indeed have the desired logical consequence.

In the following example, we do not have a logical consequence!

Example.

Let the assumptions be:

$$\neg P, P \vee Q$$

Do they entail

$$P \wedge Q?$$

We construct the truth table:

P	Q	$\neg P$	$P \vee Q$	$P \wedge Q$
$\vdots$	$\vdots$			
F	T	T	T	F
$\vdots$	$\vdots$			

We see that one row is problematic: if P is false and Q is true, then both assumptions are true, yet the desired conclusion is false. Thus $P \wedge Q$ is not a logical consequence of the assumptions $\neg P$ and $P \vee Q$.

1.5. Limitations

We have just seen that if there are n variables $P_1, \dots, P_n$ in our assumptions $\mathcal{P}_1, \dots, \mathcal{P}_l$ and conclusion Q , then we must construct a truth table with 2^n rows to verify the logical consequence.

As n grows, 2^n grows exponentially (since $2^n = e^{n \ln 2}$), and constructing such a table by hand becomes impractical. More significantly, we are still working within the fairly limited setting of true/false logic. This truth-table method will no longer be possible once we introduce the parameters of first-order logic. Consider the following problem, where $n \in \mathbb{N}$ is an integer. We will all agree that we have the following logical consequence:

$$n > 0, n \text{ is even} \models n^2 - 1 \text{ is odd}$$

But if we want to justify it with a truth table, the table has infinitely many rows depending on the parameter n , and this cannot be accepted as a proof.

n	$n > 0$	$n \text{ is even}$	$n^2 - 1 \text{ is odd}$
0	F	T	—
1	T	F	—
2	T	T	T
3	T	F	—
4	T	T	T
$\vdots$	$\vdots$	$\vdots$	$\vdots$

2. Formal proof

2.1. What is a proof?

A proof aims to convince others that a statement is true. A proof must have a finite number of steps. Anyone must be able to check each of these steps, and the transition from one step to the next must be easy to understand and require no deep thought. There is a natural similarity between an algorithm (a sequence of elementary instructions intended to be read by a machine) and a proof (intended for a human). With the advent of proof assistants, this difference is fading.

2.2. Formal proof $\vdash$

Let $\mathcal{P}_1, \dots, \mathcal{P}_l$ be formulas, which will be our assumptions, and let $\mathcal{Q}$ be a formula (the conclusion). We write

$$\boxed{\mathcal{P}_1, \dots, \mathcal{P}_l \vdash \mathcal{Q}}$$

if there is a *proof* of $\mathcal{Q}$ from $\mathcal{P}_1, \dots, \mathcal{P}_l$. This expression is then called a *theorem*. The symbol $\vdash$ is the *turnstile* and is read “proves” or “allows us to prove.”

But how do we construct a proof? It is a finite sequence of formulas that starts from the assumptions and leads to the conclusion. Each line of the proof is obtained by applying one of a small number of permitted logical rules to earlier lines.

More precisely:

Definition.

A *proof* is a finite sequence of formulas $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_N$ in which each $\mathcal{A}_i$ is:

- one of the assumptions $\mathcal{P}_1, \dots, \mathcal{P}_l$ in the statement, or
- a consequence obtained by applying the *rules of inference* to earlier lines $\mathcal{A}_1, \dots, \mathcal{A}_{i-1}$.
- And $\mathcal{A}_N$ is the conclusion $\mathcal{Q}$.

The rules of inference will be described in more detail below. For example, a rule may come from elementary logical formulas such as:

- if $\mathcal{P} \wedge \mathcal{Q}$, then $\mathcal{P}$,
- if $\mathcal{P}$ and $\mathcal{Q}$, then $\mathcal{P} \wedge \mathcal{Q}$,
- ...

Let us summarize the work of a mathematician. The mathematician states a theorem:

Theorem.

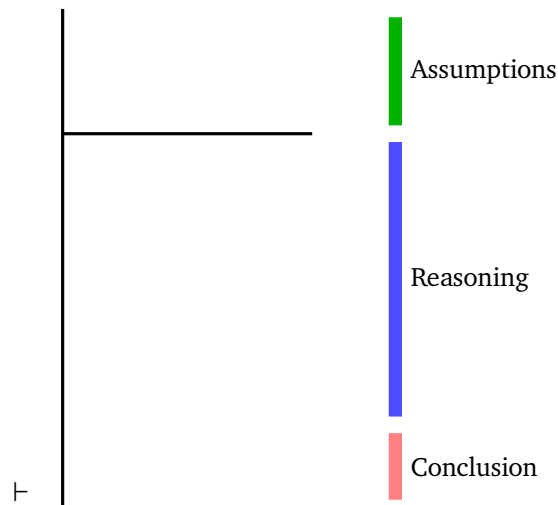
$$\mathcal{P}_1, \dots, \mathcal{P}_l \vdash \mathcal{Q}$$

Then the mathematician gives its proof.

- Remark. Here, “theorem” means a proved statement. We make no qualitative distinction between Lemma/Proposition/Theorem/Corollary according to their importance or interest.
- Warning. At first sight, no notion of true or false appears explicitly in this section on proofs. The connection with true/false logic is actually implicit: it lies in the choice of rules of inference, which were designed to respect this logic.

2.3. A first example

A *natural deduction proof* is presented as an array containing a sequence of lines: first the assumptions, then the reasoning, and finally the conclusion.



Before listing the rules of inference, here is a practical example of a statement and its proof.

Theorem.

$$\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}) \vdash (\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$$

Proof of the theorem.

$\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$	Assumption						
$\mathcal{P}$	Left-hand term of the assumption						
$\mathcal{Q} \vee \mathcal{R}$	Right-hand term of the assumption						
<table> <tr> <td>$\mathcal{Q}$</td><td>Temporary assumption. First case of OR</td></tr> <tr> <td>$\mathcal{P} \wedge \mathcal{Q}$</td><td>Because we already have $\mathcal{P}$, and here we have $\mathcal{Q}$</td></tr> <tr> <td>$(\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$</td><td>Because the left-hand term is true</td></tr> </table>	$\mathcal{Q}$	Temporary assumption. First case of OR	$\mathcal{P} \wedge \mathcal{Q}$	Because we already have $\mathcal{P}$, and here we have $\mathcal{Q}$	$(\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$	Because the left-hand term is true	
$\mathcal{Q}$	Temporary assumption. First case of OR						
$\mathcal{P} \wedge \mathcal{Q}$	Because we already have $\mathcal{P}$, and here we have $\mathcal{Q}$						
$(\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$	Because the left-hand term is true						
<table> <tr> <td>$\mathcal{R}$</td><td>Temporary assumption. Second case of OR</td></tr> <tr> <td>$\mathcal{P} \wedge \mathcal{R}$</td><td>Because we already have $\mathcal{P}$, and here we have $\mathcal{R}$</td></tr> <tr> <td>$(\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$</td><td>Because the right-hand term is true</td></tr> </table>	$\mathcal{R}$	Temporary assumption. Second case of OR	$\mathcal{P} \wedge \mathcal{R}$	Because we already have $\mathcal{P}$, and here we have $\mathcal{R}$	$(\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$	Because the right-hand term is true	
$\mathcal{R}$	Temporary assumption. Second case of OR						
$\mathcal{P} \wedge \mathcal{R}$	Because we already have $\mathcal{P}$, and here we have $\mathcal{R}$						
$(\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$	Because the right-hand term is true						
$\vdash (\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$	Conclusion. Both cases lead to the same result.						

The rules we used are the introduction and elimination rules for AND and OR:

$$\mathcal{P} \wedge \mathcal{Q} \quad \text{means} \quad \mathcal{P} \text{ and } \mathcal{Q}$$

$$\mathcal{P} \vee \mathcal{Q} \quad \text{means} \quad \mathcal{P} \text{ or } \mathcal{Q}$$

3. Rules of inference

Here we give the list of rules permitted in a proof. Some are natural, while others are a little less obvious. We begin with a set of six *basic rules* that form the foundation of proofs in classical logic. We will then present *derived rules*, which follow from the basic rules and make proofs easier to write.

3.1. The basic rules

Rule 1. Reuse.

$$\boxed{\mathcal{P} \vdash \mathcal{P}}$$

Reuse simply means that we may reuse a formula obtained earlier.

Rule 2. Conjunction.

$$\boxed{\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P} \quad \text{and} \quad \mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q}}$$

$$\boxed{\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}}$$

The rules for AND are natural. For example, if we have already obtained both $\mathcal{P}$ and $\mathcal{Q}$, then we have proved $\mathcal{P} \wedge \mathcal{Q}$.

Rule 3. Double negation.

$$\boxed{\neg\neg\mathcal{P} \vdash \mathcal{P}}$$

This rule will be discussed in a remark below.

Here is how these rules are written in a natural deduction proof:

$$\begin{array}{ccc}
 \begin{array}{|l} \vdots \\ \mathcal{P} \\ \vdots \\ \mathcal{P} \\ \vdots \end{array} & \vdash & \begin{array}{|l} \vdots \\ \mathcal{P} \wedge \mathcal{Q} \\ \vdots \\ \mathcal{P} \end{array} \\
 & & \vdash \quad \begin{array}{|l} \vdots \\ \mathcal{P} \wedge \mathcal{Q} \end{array}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{|l} \vdots \\ \mathcal{P} \\ \vdots \\ \mathcal{Q} \\ \vdots \end{array} & \vdash & \begin{array}{|l} \vdots \\ \neg\neg\mathcal{P} \\ \vdots \\ \mathcal{P} \end{array}
 \end{array}$$

When writing a proof, we may use any formula obtained earlier. We can, of course, apply several rules in succession before reaching the desired final conclusion, but each line corresponds to the application of a single rule.

Rule 4. Disjunction.

$$\boxed{\mathcal{P} \vdash \mathcal{P} \vee \mathcal{Q} \quad \text{and} \quad \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}}$$

The converse direction is harder to write formally: “ $\mathcal{P} \vee \mathcal{Q}$ proves $\mathcal{P}$ or $\mathcal{Q}$.” To express it correctly, we introduce an auxiliary formula $\mathcal{R}$ as the conclusion of implications:

$$\boxed{\mathcal{P} \vee \mathcal{Q}, \mathcal{P} \rightarrow \mathcal{R}, \mathcal{Q} \rightarrow \mathcal{R} \quad \vdash \quad \mathcal{R}}$$

In practice, this is fairly easy (the array on the right below): if we have $\mathcal{P} \vee \mathcal{Q}$, we consider two cases. In the first case, we assume $\mathcal{P}$; in the second, we assume $\mathcal{Q}$. If we prove the same conclusion $\mathcal{R}$ in both cases, then $\mathcal{R}$ is proved.

$$\begin{array}{c} \vdots \\ \mathcal{P} \\ \vdots \\ \vdash \quad \mathcal{P} \vee \mathcal{Q} \end{array} \qquad \begin{array}{c} \mathcal{P} \vee \mathcal{Q} \\ \hline \mathcal{P} \qquad \text{First case of OR. Temporary assumption } \mathcal{P} \\ \hline \mathcal{R} \\ \hline \mathcal{Q} \qquad \text{Second case of OR. Temporary assumption } \mathcal{Q} \\ \hline \mathcal{R} \\ \vdash \quad \mathcal{R} \end{array}$$

Rule 5. Implication.

$$\boxed{\mathcal{P}, \mathcal{P} \rightarrow \mathcal{Q} \quad \vdash \quad \mathcal{Q}}$$

This rule is called *modus ponens*. It is a kind of elementary mini-proof: if we have an assumption $\mathcal{P}$ and know that $\mathcal{P}$ implies $\mathcal{Q}$, then we have proved $\mathcal{Q}$!

The converse operation, namely introducing the implication, is harder to express:

$$\boxed{\text{If } \mathcal{P} \vdash \mathcal{Q} \quad \text{then} \quad \vdash \mathcal{P} \rightarrow \mathcal{Q}.}$$

If we can prove $\mathcal{Q}$ under the assumption $\mathcal{P}$, then we have a proof of $\mathcal{P} \rightarrow \mathcal{Q}$.

Fortunately, this second rule is again easy to understand and use with a proof array (the figure on the right below).

$$\begin{array}{c} \vdots \\ \mathcal{P} \\ \vdots \\ \mathcal{P} \rightarrow \mathcal{Q} \\ \vdots \\ \vdash \quad \mathcal{Q} \end{array} \qquad \begin{array}{c} \mathcal{P} \qquad \text{Temporary assumption} \\ \hline \mathcal{Q} \qquad \text{Conclusion under this assumption} \\ \hline \vdash \quad \mathcal{P} \rightarrow \mathcal{Q} \end{array}$$

Rule 6. Explosion.

$$\boxed{\perp \quad \vdash \quad \mathcal{P}}$$

The rule of explosion (*ex falso*) says: “False implies anything.” This may seem a little strange, but it will be useful, for example, in proofs by contradiction.

Remark.

- We will state other rules below, but these rules can be derived from the basic rules.
- For each of the connectives $\wedge$, $\vee$, $\rightarrow$, there are two associated rules: an *elimination* rule that removes the symbol (for example, $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P}$) and an *introduction* rule that introduces the symbol (for example, $\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$).
- The basic rules just stated correspond to what is called *classical logic*. These are conventional choices that agree with our mathematical experience, but other approaches are possible.
- *Minimal logic* rejects the rules $\neg\neg\mathcal{P} \vdash \mathcal{P}$ and $\perp \vdash \mathcal{P}$. *Intuitionistic logic* rejects only $\neg\neg\mathcal{P} \vdash \mathcal{P}$.
- Why reject double-negation elimination? This rule is in fact equivalent to the law of excluded middle, which says that we always have $\mathcal{P} \vee \neg\mathcal{P}$. When we include the rule $\neg\neg\mathcal{P} \vdash \mathcal{P}$, we allow the following kind of proof: “If $\neg\mathcal{P}$ cannot occur, then we have a proof of $\mathcal{P}$.” Intuitionists object to this kind of reasoning because it does not provide a constructive proof.
- The rule of explosion $\perp \vdash \mathcal{P}$ is excluded from some kinds of logic because, on the one hand, it does not give a constructive proof of $\mathcal{P}$ and, on the other, as soon as a contradiction (even a local one) is found, the rule makes it possible to prove any formula, making all reasoning trivial and therefore unusable.

3.2. Examples**Example.**

$(\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{R} \vdash \mathcal{Q} \vee \mathcal{R}$		
$(\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{R}$		Assumption
$\mathcal{P} \wedge \mathcal{Q}$		First case of OR
$\mathcal{Q}$		By elimination in $\mathcal{P} \wedge \mathcal{Q}$
$\mathcal{Q} \vee \mathcal{R}$		Since we have $\mathcal{Q}$
$\mathcal{R}$		Second case of OR
$\mathcal{Q} \vee \mathcal{R}$		Since we have $\mathcal{R}$
$\vdash \mathcal{Q} \vee \mathcal{R}$		Both cases give $\mathcal{Q} \vee \mathcal{R}$

Example.

$$\mathcal{P} \vee \mathcal{Q}, \mathcal{P} \rightarrow \mathcal{R}, \mathcal{Q} \rightarrow \mathcal{S} \vdash \mathcal{R} \vee \mathcal{S}$$

	$\mathcal{P} \vee \mathcal{Q}$	Assumptions
	$\mathcal{P} \rightarrow \mathcal{R}$	
	$\mathcal{Q} \rightarrow \mathcal{S}$	
	$\mathcal{P}$	First case of OR
	$\mathcal{R}$	We apply $\mathcal{P} \rightarrow \mathcal{R}$
	$\mathcal{R} \vee \mathcal{S}$	
	$\mathcal{Q}$	Second case of OR
	$\mathcal{S}$	
	$\mathcal{R} \vee \mathcal{S}$	
$\vdash$	$\mathcal{R} \vee \mathcal{S}$	Both cases lead to the same conclusion

Example.

$$\mathcal{P} \rightarrow \mathcal{Q}, \mathcal{P} \rightarrow \mathcal{R} \quad \vdash \quad \mathcal{P} \rightarrow (\mathcal{Q} \wedge \mathcal{R})$$

	$\mathcal{P} \rightarrow \mathcal{Q}$	Assumptions
	$\mathcal{P} \rightarrow \mathcal{R}$	
	$\mathcal{P}$	Temporary assumption
	$\mathcal{Q}$	Since we have $\mathcal{P}$ and $\mathcal{P} \rightarrow \mathcal{Q}$
	$\mathcal{R}$	Since we have $\mathcal{P}$ and $\mathcal{P} \rightarrow \mathcal{R}$
	$\mathcal{Q} \wedge \mathcal{R}$	
$\vdash$	$\mathcal{P} \rightarrow (\mathcal{Q} \wedge \mathcal{R})$	Whenever we have $\mathcal{P}$, we also have $\mathcal{Q} \wedge \mathcal{R}$

Example.

$$\mathcal{P} \vee \mathcal{Q}, \neg \mathcal{P} \quad \vdash \quad \mathcal{Q}$$

This is an example of the use of explosion: once we have proved $\perp$, we can prove anything.

	$\mathcal{P} \vee \mathcal{Q}$	
	$\neg \mathcal{P}$	
	$\mathcal{P}$	First case of OR
	$\perp$	Contradiction: we have $\mathcal{P}$ and $\neg \mathcal{P}$
	$\mathcal{Q}$	Anything follows by explosion
	$\mathcal{Q}$	Second case of OR
	$\mathcal{Q}$	
$\vdash$	$\mathcal{Q}$	Conclusion from both cases

3.3. Derived rules

From the basic rules of inference, we derive a limited number of other rules. These new rules make proofs easier to write.

Derived rules. Equivalence.

As expected, we prove an equivalence by proving both implications, and conversely.

$$\boxed{\mathcal{P} \leftrightarrow \mathcal{Q} \vdash \mathcal{P} \rightarrow \mathcal{Q} \quad \text{and} \quad \mathcal{P} \leftrightarrow \mathcal{Q} \vdash \mathcal{Q} \rightarrow \mathcal{P}}$$

$$\boxed{\mathcal{P} \rightarrow \mathcal{Q}, \mathcal{Q} \rightarrow \mathcal{P} \vdash \mathcal{P} \leftrightarrow \mathcal{Q}}$$

Derived rules. Negation.

First, here is the definition of negation in terms of implication and the always-false formula $\perp$.

$$\boxed{\neg \mathcal{P} \text{ is defined as } \mathcal{P} \rightarrow \perp}$$

Thus, by definition, we have the two rules:

$$\boxed{\neg \mathcal{P} \vdash \mathcal{P} \rightarrow \perp \quad \text{and} \quad \mathcal{P} \rightarrow \perp \vdash \neg \mathcal{P}}$$

Thus, if we have $\mathcal{P}$ and $\neg \mathcal{P}$, then implication gives us a contradiction:

$$\boxed{\mathcal{P}, \neg \mathcal{P} \vdash \perp}$$

Of course, $\neg \top$ is $\perp$, and conversely $\neg \perp$ is $\top$.

We also have the converse direction of the double-negation rule:

$$\boxed{\mathcal{P} \vdash \neg \neg \mathcal{P}}$$

We also have the law of excluded middle:

$$\boxed{\vdash \mathcal{P} \vee \neg \mathcal{P}}$$

That is, without any assumption, we always have a proof of $\mathcal{P} \vee \neg\mathcal{P}$.

Derived rules. Conjunction/disjunction.

For example:

$$\boxed{\mathcal{P} \vee \mathcal{Q}, \neg\mathcal{Q} \vdash \mathcal{P}}$$

We can also consider all the proofs related to De Morgan's laws, for example:

$$\boxed{\neg(\mathcal{P} \wedge \mathcal{Q}) \vdash \neg\mathcal{P} \vee \neg\mathcal{Q}}$$

We will see how to derive this rule from the basic rules in the examples below.

Derived rules. Implication.

First, here is the rule called *modus tollens*:

$$\boxed{\mathcal{P} \rightarrow \mathcal{Q}, \neg\mathcal{Q} \vdash \neg\mathcal{P}}$$

If $\mathcal{P}$ implies $\mathcal{Q}$, but we do not have $\mathcal{Q}$, then we have a proof of $\neg\mathcal{P}$. The examples below explain how to derive this rule from the basic rules.

The law of excluded middle allows a proof by cases:

$$\boxed{\mathcal{P} \rightarrow \mathcal{Q}, \neg\mathcal{P} \rightarrow \mathcal{Q} \vdash \mathcal{Q}}$$

This allows us to reason using the two cases of $\mathcal{P} \vee \neg\mathcal{P}$ (whatever the formula $\mathcal{P}$).

$$\begin{array}{c|c|l} & \mathcal{P} & \text{First case} \\ & \hline & \mathcal{Q} & \\ & \neg\mathcal{P} & \text{Second case} \\ & \hline & \mathcal{Q} & \\ \vdash & \mathcal{Q} & \end{array}$$

Derived rules. Proof by contradiction and contraposition.

Proof by contradiction:

$$\boxed{\neg\mathcal{P} \rightarrow \perp \vdash \mathcal{P}}$$

If $\neg\mathcal{P}$ leads to a contradiction, then we have proved $\mathcal{P}$.

Contraposition:

$$\boxed{\mathcal{P} \rightarrow \mathcal{Q} \vdash \neg\mathcal{Q} \rightarrow \neg\mathcal{P}}$$

Or, equivalently:

$$\boxed{\neg\mathcal{Q} \rightarrow \neg\mathcal{P} \vdash \mathcal{P} \rightarrow \mathcal{Q}}$$

3.4. Examples

Example.

		$\mathcal{P} \vee \mathcal{Q} \rightarrow \mathcal{P} \wedge \mathcal{Q} \quad \vdash \quad \mathcal{P} \leftrightarrow \mathcal{Q}$
$\vdash$	$\mathcal{P} \vee \mathcal{Q} \rightarrow \mathcal{P} \wedge \mathcal{Q}$	
	$\mathcal{P}$	Assume $\mathcal{P}$
	$\mathcal{P} \vee \mathcal{Q}$	
	$\mathcal{P} \wedge \mathcal{Q}$	By the initial assumption
	$\mathcal{Q}$	
	$\mathcal{P} \rightarrow \mathcal{Q}$	First direction of the equivalence
	$\mathcal{Q}$	Assume $\mathcal{Q}$
	$\mathcal{P} \vee \mathcal{Q}$	
	$\mathcal{P} \wedge \mathcal{Q}$	By the implication in the initial assumption
	$\mathcal{P}$	
	$\mathcal{Q} \rightarrow \mathcal{P}$	Second direction of the equivalence
	$\mathcal{P} \leftrightarrow \mathcal{Q}$	

Example.

Let us prove *modus tollens* directly by contradiction:

		$\mathcal{P} \rightarrow \mathcal{Q}, \neg \mathcal{Q} \quad \vdash \quad \neg \mathcal{P}$
$\vdash$	$\mathcal{P} \rightarrow \mathcal{Q}$	
	$\neg \mathcal{Q}$	
	$\mathcal{P}$	For a contradiction, assume $\mathcal{P}$
	$\mathcal{Q}$	The initial assumption gives $\mathcal{Q}$
	$\perp$	Contradiction: we have $\mathcal{Q}$ and $\neg \mathcal{Q}$
	$\mathcal{P} \rightarrow \perp$	
	$\neg \mathcal{P}$	

Example.

Let us prove the following De Morgan rule directly:

$$\neg(\mathcal{P} \wedge \mathcal{Q}) \quad \vdash \quad \neg \mathcal{P} \vee \neg \mathcal{Q}$$

	$\neg(\mathcal{P} \wedge \mathcal{Q})$	
	$\mathcal{P}$	First case: assume $\mathcal{P}$
	$\mathcal{Q}$	For a contradiction, assume $\mathcal{Q}$
	$\mathcal{P} \wedge \mathcal{Q}$	
	$\perp$	Contradiction, obtained from the initial assumption
	$\neg \mathcal{Q}$	
	$\neg \mathcal{P} \vee \neg \mathcal{Q}$	
	$\neg \mathcal{P}$	Second case: assume $\neg \mathcal{P}$
	$\neg \mathcal{P} \vee \neg \mathcal{Q}$	
$\vdash$	$\neg \mathcal{P} \vee \neg \mathcal{Q}$	Conclusion obtained in both cases of excluded middle

3.5. Proof by contradiction and contraposition

In classical logic, to prove an implication $\mathcal{P} \rightarrow \mathcal{Q}$, we can replace a proof by contraposition with a proof by contradiction, and conversely.

The two arguments in fact use essentially the same steps.

From contraposition to contradiction.

Suppose that we know how to prove the contrapositive $\neg \mathcal{Q} \rightarrow \neg \mathcal{P}$. Such a proof has the form

$$\neg \mathcal{Q} \longrightarrow \boxed{\mathcal{C}} \longrightarrow \neg \mathcal{P},$$

where $\mathcal{C}$ denotes the intermediate steps of the argument (the proof on the left below).

To obtain a proof by contradiction of $\mathcal{P} \rightarrow \mathcal{Q}$, we assume $\mathcal{P}$ and then $\neg \mathcal{Q}$. We repeat exactly the same argument $\mathcal{C}$, which yields $\neg \mathcal{P}$, contradicting $\mathcal{P}$ (the proof on the right below).

	$\neg \mathcal{Q}$			$\mathcal{P} \wedge \neg \mathcal{Q}$	
	$\boxed{\mathcal{C}}$	Intermediate steps		$\boxed{\mathcal{C}}$	Same steps
	$\neg \mathcal{P}$			$\mathcal{P} \wedge \neg \mathcal{P}$	
$\vdash$			$\vdash$	$\perp$	

From contradiction to contraposition.

Now suppose that we have a proof by contradiction of $\mathcal{P} \rightarrow \mathcal{Q}$ (on the left below). It has the form:

$$\mathcal{P}, \neg \mathcal{Q} \longrightarrow \boxed{\mathcal{C}'} \longrightarrow \perp.$$

To obtain the contrapositive, we assume $\neg \mathcal{Q}$ and seek to prove $\neg \mathcal{P}$. We therefore assume $\mathcal{P}$ and repeat exactly the same argument $\mathcal{C}'$, which leads to a contradiction (on the right below).

$\frac{\begin{array}{c c} P \wedge \neg Q \\ \hline \boxed{C'} \\ \hline \perp \end{array}}{\vdash}$	Intermediate steps	$\frac{\begin{array}{c c c} \neg Q & & \\ \hline & P & \\ \hline & \boxed{C'} & \\ \hline & \perp & \\ \hline \neg P \end{array}}{\vdash}$	Same steps
--	--------------------	--	------------

Thus, a proof by contraposition and a proof by contradiction have the same core: only the arrangement of the assumptions changes.

3.6. Summary

Here are the main rules of inference used in this chapter, with the basic rules of classical logic in bold.

Name	Rule
Reuse	$P \vdash P$
$\wedge$ elimination	$P \wedge Q \vdash P$ $P \wedge Q \vdash Q$
$\wedge$ introduction	$P, Q \vdash P \wedge Q$
Double-negation elimination	$\neg\neg P \vdash P$
$\vee$ introduction	$P \vdash P \vee Q$ $Q \vdash P \vee Q$
$\vee$ elimination	$P \vee Q, P \rightarrow R, Q \rightarrow R \vdash R$
Implication (modus ponens)	$P, P \rightarrow Q \vdash Q$
$\rightarrow$ introduction	If $P \vdash Q$ then $\vdash P \rightarrow Q$
Explosion (ex falso)	$\perp \vdash P$
<i>Equivalence</i>	$P \leftrightarrow Q \vdash P \rightarrow Q$ $P \leftrightarrow Q \vdash Q \rightarrow P$ $P \rightarrow Q, Q \rightarrow P \vdash P \leftrightarrow Q$
<i>Definition of negation</i>	$\neg P$ is defined as $P \rightarrow \perp$
<i>Double-negation introduction</i>	$P \vdash \neg\neg P$
<i>Law of excluded middle</i>	$\vdash P \vee \neg P$
<i>Modus tollens</i>	$P \rightarrow Q, \neg Q \vdash \neg P$
<i>De Morgan's laws</i>	$\neg(P \wedge Q) \vdash \neg P \vee \neg Q$...
<i>Proof by contradiction</i>	$\neg P \rightarrow \perp \vdash P$
<i>Contrapositive</i>	$P \rightarrow Q \vdash \neg Q \rightarrow \neg P$ $\neg Q \rightarrow \neg P \vdash P \rightarrow Q$

4. An overview of Gödel's Completeness Theorem

Gödel's *Incompleteness Theorem* (1931) is the ultimate goal of this book. This theorem will shake the foundations of logic. But to truly grasp its importance, we must first understand the *Completeness Theorem* (1929), which is rather reassuring. Its statement even seems ordinary, because it is exactly what everyone would expect.

Theorem 1 (Completeness Theorem).

For true/false logic:

$$\mathcal{P} \models \mathcal{Q} \quad \text{if and only if} \quad \mathcal{P} \vdash \mathcal{Q}.$$

The key point is that we are working here in the framework of propositional logic, presented in the chapter “True/False Logic.” This is also true in the more general framework of first-order logic (see the chapter “First-Order Logic”). Indeed, that is the framework in which we will prove this theorem (see the chapter “Completeness Theorem”).

Let us examine each implication in the equivalence stated above.

Reverse direction $\Leftarrow$. $\mathcal{P} \vdash \mathcal{Q}$ implies $\mathcal{P} \models \mathcal{Q}$.

This implication is called the *Soundness Theorem*; it is a statement of “common sense.” If we have a proof that $\mathcal{P}$ entails $\mathcal{Q}$ and $\mathcal{P}$ is true, then $\mathcal{Q}$ is true. In other words, when we have proved a theorem and its assumptions are true, then its conclusion is true! Note once again that the formal definition of a proof contains no concept of true or false. The Soundness Theorem means that the rules of inference chosen to define a proof are coherent and compatible with the rules of propositional logic.

Forward direction $\Rightarrow$. $\mathcal{P} \models \mathcal{Q}$ implies $\mathcal{P} \vdash \mathcal{Q}$.

This direction is the *Completeness Theorem*. It means that for every statement we find to be true, we can find a proof that makes it a theorem. Thus, if we find that whenever $\mathcal{P}$ is true, $\mathcal{Q}$ is also true, then there is a proof that $\mathcal{P}$ entails $\mathcal{Q}$. This result holds within propositional logic (our true/false logic).

This gives us sweet dreams: in propositional logic, everything that is true is provable, and conversely. The nightmare will begin later in this book...

Exercises

Proof by truth table

Exercise 1. Determine whether the following logical consequences are valid. Justify your answer.

$(\mathcal{P} \wedge \neg \mathcal{Q}) \vee (\neg \mathcal{P} \wedge \mathcal{Q})$	$\stackrel{?}{\models}$	$\mathcal{P} \vee \mathcal{Q}$
$\mathcal{P} \rightarrow \mathcal{Q}, \neg \mathcal{P}$	$\stackrel{?}{\models}$	$\neg \mathcal{Q}$
$\mathcal{P} \rightarrow \mathcal{Q}, \mathcal{R} \rightarrow \mathcal{Q}$	$\stackrel{?}{\models}$	$\mathcal{P} \rightarrow \mathcal{R}$
$\mathcal{P} \rightarrow \neg \mathcal{Q}$	$\stackrel{?}{\models}$	$\mathcal{Q} \rightarrow \neg \mathcal{P}$
$\mathcal{P} \rightarrow \mathcal{Q}, \mathcal{P} \vee \neg \mathcal{Q}$	$\stackrel{?}{\models}$	$\mathcal{Q}$
$\mathcal{P} \vee \neg \mathcal{Q}, \mathcal{P} \vee \neg \mathcal{R}, \mathcal{R} \vee \neg \mathcal{Q}$	$\stackrel{?}{\models}$	$\neg \mathcal{P} \rightarrow \mathcal{Q} \wedge \mathcal{R}$

Formal proof

Exercise 2. Explain the following theorems in English:

$$\begin{array}{rcl}
 \mathcal{P}, \mathcal{Q} & \vdash & \mathcal{P} \wedge \mathcal{Q} \\
 \mathcal{P} \wedge (\mathcal{P} \rightarrow \mathcal{Q}) & \vdash & \mathcal{Q} \\
 \mathcal{P} \rightarrow \mathcal{Q}, \mathcal{Q} \rightarrow \mathcal{R} & \vdash & \mathcal{P} \rightarrow \mathcal{R} \\
 \neg \mathcal{Q} \rightarrow \neg \mathcal{P} & \vdash & \mathcal{P} \rightarrow \mathcal{Q} \\
 & \vdash & \mathcal{P} \vee \neg \mathcal{P}
 \end{array}$$

Exercise 3. Consider the following theorem:

$$\mathcal{P} \vee \mathcal{Q}, \mathcal{P} \rightarrow \mathcal{R}, \mathcal{Q} \rightarrow \mathcal{S} \quad \vdash \quad \mathcal{R} \vee \mathcal{S}$$

Complete the missing lines of the proof and comment on every line.

	...	Assumption 1
	...	...
	...	...
	$\mathcal{P}$ 	Temporary assumption. First case of OR We used the implication ... Intermediate conclusion Temporary assumption.
$\vdash$	...	Conclusion because ...

Rules of inference

Exercise 4. Give natural deduction proofs of the following theorems:

$$\begin{array}{rcl}
 \mathcal{P} \rightarrow \mathcal{Q}, \mathcal{Q} \rightarrow \mathcal{R} & \vdash & \mathcal{P} \rightarrow \mathcal{R} \\
 \mathcal{P} & \vdash & (\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \neg \mathcal{Q}) \\
 (\mathcal{P} \rightarrow \mathcal{Q}) \rightarrow \mathcal{R} & \vdash & \mathcal{P} \rightarrow (\mathcal{Q} \rightarrow \mathcal{R}) \\
 \mathcal{P} \leftrightarrow \mathcal{Q} & \vdash & (\mathcal{P} \vee \mathcal{Q}) \rightarrow (\mathcal{P} \wedge \mathcal{Q}) \\
 \neg \mathcal{P} \vee \neg \mathcal{Q} & \vdash & \neg(\mathcal{P} \wedge \mathcal{Q})
 \end{array}$$

Exercise 5. Give natural deduction proofs of the following theorems:

$$\begin{array}{rcl}
 \neg \mathcal{B} \rightarrow (\mathcal{A} \vee \mathcal{B}) & \vdash & \mathcal{A} \\
 \mathcal{C} \rightarrow (\mathcal{D} \rightarrow \mathcal{E}) & \vdash & (\mathcal{C} \wedge \mathcal{D}) \rightarrow \mathcal{E} \\
 \mathcal{F} \leftrightarrow \mathcal{G}, \mathcal{G} \leftrightarrow \mathcal{H} & \vdash & \mathcal{F} \leftrightarrow \mathcal{H} \\
 & \vdash & (\mathcal{I} \rightarrow \mathcal{J}) \vee (\mathcal{J} \rightarrow \mathcal{I}) \\
 \mathcal{K} \vee \mathcal{L}, \mathcal{L} \vee \mathcal{M}, \neg \mathcal{L} & \vdash & \mathcal{K} \wedge \mathcal{M}
 \end{array}$$

Exercise 6. The aim of this exercise is to show that the law of excluded middle, “ $\vdash \mathcal{P} \vee \neg \mathcal{P}$ ” (with no assumptions), follows from the basic rules. Recall that, by definition, $\neg \mathcal{P}$ means $\mathcal{P} \rightarrow \perp$.

1. Check that $\neg \neg \mathcal{Q}$ can be written $(\mathcal{Q} \rightarrow \perp) \rightarrow \perp$.

2. Here is the proof of:

$$\vdash ((\mathcal{P} \vee \neg \mathcal{P}) \rightarrow \perp) \rightarrow \perp$$

$$\begin{array}{c}
 \vdash \left| \begin{array}{c}
 \overline{\emptyset} \\
 (\mathcal{P} \vee \neg \mathcal{P}) \rightarrow \perp \\
 \left| \begin{array}{c}
 \mathcal{P} \\
 \hline
 \mathcal{P} \vee \neg \mathcal{P} \\
 \perp \\
 \mathcal{P} \rightarrow \perp \\
 \neg \mathcal{P} \\
 \mathcal{P} \vee \neg \mathcal{P} \\
 \perp
 \end{array}
 \right. \\
 ((\mathcal{P} \vee \neg \mathcal{P}) \rightarrow \perp) \rightarrow \perp
 \end{array} \right.
 \end{array}$$

Explain each line and check that only the basic rules are used.

3. Conclude using a basic rule.

Exercise 7. The aim is to obtain derived rules from the basic rules.

1. Prove the derived rule

$$\mathcal{P} \vdash \neg \neg \mathcal{P}$$

using only the basic rules. Recall that $\neg \mathcal{P}$ means $\mathcal{P} \rightarrow \perp$.

2. Here we allow the basic rules and the law of excluded middle (which we derived from the basic rules). We may therefore reason by distinguishing the cases $\mathcal{P}$ and $\neg \mathcal{P}$. Prove the rule for proof by contradiction:

$$\neg \mathcal{P} \rightarrow \perp \vdash \mathcal{P}$$

3. Deduce that this form of contraposition can be derived using only the basic rules:

$$\mathcal{P} \rightarrow \mathcal{Q} \vdash \neg \mathcal{Q} \rightarrow \neg \mathcal{P}$$

An overview of Gödel's Completeness Theorem

Exercise 8. Show that both

$$\neg(\mathcal{P} \vee \mathcal{Q}) \models (\neg \mathcal{P}) \wedge (\neg \mathcal{Q})$$

and

$$\neg(\mathcal{P} \vee \mathcal{Q}) \vdash (\neg \mathcal{P}) \wedge (\neg \mathcal{Q})$$

hold.

Exercise 9. Show that both

$$\neg \mathcal{P} \vee \mathcal{Q}, \neg \mathcal{P} \rightarrow \mathcal{R}, \mathcal{Q} \rightarrow \mathcal{R} \models \mathcal{R}$$

and

$$\neg \mathcal{P} \vee \mathcal{Q}, \neg \mathcal{P} \rightarrow \mathcal{R}, \mathcal{Q} \rightarrow \mathcal{R} \vdash \mathcal{R}$$

hold.