

We adapt the construction of proofs to the quantifiers of first-order logic.

1. Proofs in first-order logic

1.1. Review

Recall that a proof consists in starting from a set of formulas $\mathcal{P}_1, \dots, \mathcal{P}_l$ and arriving at a formula $\mathcal{Q}$ in a finite number of steps by following precise rules. If we have obtained a proof, we write:

$$\mathcal{P}_1, \dots, \mathcal{P}_l \vdash \mathcal{Q}$$

For first-order logic, we first retain all the rules of inference for proofs in true/false logic. We refer to the chapter “Proofs” in Part One. We will add new rules involving quantifiers. Recall that there is still no connection between what we prove and what is true. This connection will be established in the following chapters. In particular, in the next chapter we will show that everything that is proved is true. In this chapter, we fix a language $\mathcal{L}$. A proof involves only this language. There are no structures or assignments to consider. Thus, the formulas in a proof remain in $\mathcal{L}$, which is one respect in which the notion of proof ($\vdash$) is simpler than the notion of satisfaction ($\models$).

1.2. Rules for quantifiers

We present four new rules involving quantifiers: the introduction and elimination rules for $\forall$ and $\exists$. An additional rule concerns equality. We present each rule briefly before returning to it in detail in the next section. There are two easy rules and two difficult rules. The difficulty lies not in the boxed rule, but rather in its conditions of application.

To clarify the notation for the rules, when we have a formula $\mathcal{P}$ and are interested in its dependence on the variable x , we naturally denote it by $\mathcal{P}(x)$. Note that it may also depend on other variables. For a formula $\mathcal{P}(x)$, we will write $\mathcal{P}(t)$ instead of $\mathcal{P}[t/x]$, the substitution of the term t for the variable x .

Rule I. $\forall$ elimination.

$\forall x \mathcal{P}(x) \vdash \mathcal{P}(c)$
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- *Conditions.* c may be any constant in the language; c may also be a *closed term*, that is, a term with no free variable.
- *Remark.* Thus, under the assumption $\forall x \mathcal{P}(x)$, we can replace every free occurrence of x by c , and this proves $\mathcal{P}(c)$ (which is in fact $\mathcal{P}[c/x]$). Here it does not matter if the constant (or term) c has already appeared earlier in the proof.

- *Example.* For any constant c :

$$\forall x \exists y P(x) \wedge Q(x, y) \vdash \exists y P(c) \wedge Q(c, y)$$

Rule II. $\forall$ introduction.

$$\boxed{\mathcal{P}(c) \vdash \forall x \mathcal{P}(x)}$$

- *Conditions.* c denotes a constant of $\mathcal{L}$, but it must be *arbitrary*, that is, it must not appear in any assumption currently in use.
- *Remark.* This is a tricky rule because here c does not really play the role of a constant, but rather that of a *free variable*. It is best to use a name c that does not appear at all in the preceding lines of the proof.
- *Example.* Let c be arbitrary:

$$c^2 \geq 0 \vdash \forall x x^2 \geq 0$$

Indeed, if we have proved $c^2 \geq 0$ for an arbitrary c , then $x^2 \geq 0$ for every x .

Rule III. $\exists$ introduction.

$$\boxed{\mathcal{P}(c) \vdash \exists x \mathcal{P}(x)}$$

- *Conditions.* c is a constant in the language or a term with no free variable.
- *Remark.* c is called a *witness* for the quantifier $\exists$. The symbol c may have appeared earlier in the proof.
- *Example.*

$$c^2 \neq c \vdash \exists x x^2 \neq x$$

Rule IV. $\exists$ elimination.

$$\boxed{\exists x \mathcal{P}(x), [\mathcal{P}(c) \vdash \mathcal{Q}] \vdash \mathcal{Q}}$$

The notation in brackets corresponds to a subproof. This is clearer when written in tabular form:

$$\begin{array}{c|c|c|c} & \exists x \mathcal{P}(x) & & \\ \hline & & \mathcal{P}(c) & \text{Temporary assumption} \\ & & \vdots & \\ & & \mathcal{Q} & \\ \hline \vdash & & \mathcal{Q} & \end{array}$$

- *Conditions.* c must be arbitrary: it must appear neither in $\mathcal{Q}$ nor in $\exists x \mathcal{P}(x)$, nor in any assumption currently in use. It is called a *fresh constant* (or, by abuse of terminology, a *fresh variable*).
- *Remark.* The condition is not obvious at first, but the principle is natural: if we know that there is an element satisfying $\mathcal{P}$, and assuming $\mathcal{P}(c)$ for an arbitrary c leads to $\mathcal{Q}$, then the conclusion $\mathcal{Q}$ is proved.

- *Example.*

$$\exists x \, f(x) = 0, \forall x \, (f(x) = 0 \rightarrow g(x) = 1) \vdash \exists x \, g(x) = 1$$

This is an almost direct application of Rule IV; see the next section for details.

Rule V. Equality.

$$\boxed{t = s, \mathcal{P}(t) \vdash \mathcal{P}(s)}$$

- *Conditions.* t and s are terms in the language. The substitutions of t or s must of course be legitimate (with no capture).
- *Remark.* There is no trap with equality. Two equal things are considered identical and therefore freely interchangeable.
- *Example.*

$$a = x + 1, a > 7 \vdash x + 1 > 7$$

1.3. Examples of proofs

Example.

$$\forall x \, \mathcal{P}(x) \vdash \exists x \, \mathcal{P}(x)$$

	$\forall x \, \mathcal{P}(x)$	Assumption
	$\mathcal{P}(c)$	$\forall$ elimination
$\vdash$	$\exists x \, \mathcal{P}(x)$	$\exists$ introduction

The proof rules implicitly assume that we are working with nonempty sets. This is consistent with the domain of a structure, which must be a nonempty set.

Example.

Let $\mathcal{P}$ be a formula in which the variable x does not appear.

$$\forall x \, (\mathcal{P} \rightarrow \mathcal{Q}(x)) \vdash \mathcal{P} \rightarrow (\forall x \, \mathcal{Q}(x))$$

1	$\forall x \, (\mathcal{P} \rightarrow \mathcal{Q}(x))$	
2	$\mathcal{P}$	Temporary assumption
3	$\mathcal{P} \rightarrow \mathcal{Q}(c)$	$\forall$ elimination from 1
4	$\mathcal{Q}(c)$	
5	$\forall x \, \mathcal{Q}(x)$	$\forall$ introduction; c appears in no assumption
$\vdash$	$\mathcal{P} \rightarrow (\forall x \, \mathcal{Q}(x))$	

Example.

$$\forall x (\mathcal{P}(x) \rightarrow \mathcal{Q}(x)), \exists x \mathcal{P}(x) \vdash \exists x \mathcal{Q}(x)$$

This is quite natural: if $\mathcal{P}(x)$ always implies $\mathcal{Q}(x)$ and one of the $\mathcal{P}(x)$ is true, then one of the $\mathcal{Q}(x)$ is true. This is a variant of *modus ponens* in propositional logic: $\mathcal{P}, \mathcal{P} \rightarrow \mathcal{Q} \vdash \mathcal{Q}$.

1	$\forall x (\mathcal{P}(x) \rightarrow \mathcal{Q}(x))$	
2	$\exists x \mathcal{P}(x)$	
3	$\mathcal{P}(c)$	Temporary assumption
4	$\mathcal{P}(c) \rightarrow \mathcal{Q}(c)$	$\forall$ elimination from 1
5	$\mathcal{Q}(c)$	
6	$\exists x \mathcal{Q}(x)$	$\exists$ introduction
$\vdash$	$\exists x \mathcal{Q}(x)$	$\exists$ elimination from 2 via 3–6

Example.

$$\exists x \forall y \mathcal{P}(x, y) \vdash \forall y \exists x \mathcal{P}(x, y)$$

We can therefore switch an $\exists \forall$ to a $\forall \exists$ (but not the other way around).

1	$\exists x \forall y \mathcal{P}(x, y)$	
2	$\forall y \mathcal{P}(a, y)$	With a arbitrary
3	$\mathcal{P}(a, b)$	With b arbitrary
4	$\exists x \mathcal{P}(x, b)$	$\exists$ introduction
5	$\forall y (\exists x \mathcal{P}(x, y))$	$\forall$ introduction
$\vdash$	$\forall y \exists x \mathcal{P}(x, y)$	$\exists$ elimination from 1 via 2–5

2. Negation of quantifiers and derived rules

2.1. Negation of quantifiers

Derived rule. Negation of quantifiers.

$$\neg (\forall x \mathcal{P}(x)) \vdash \exists x (\neg \mathcal{P}(x))$$

$$\neg (\exists x \mathcal{P}(x)) \vdash \forall x (\neg \mathcal{P}(x))$$

We are not surprised: the negation of a “for every” is a “there exists”, and conversely.

Examples:

1. $\neg (\forall x \ f(x) = g(x)) \vdash \exists x \ (f(x) \neq g(x))$
2. $\neg (\exists x \ P(x) \wedge Q(x)) \vdash \forall x \ (\neg P(x)) \vee (\neg Q(x))$
3. $\neg (\forall x \ P(x) \rightarrow Q(x)) \vdash \exists x \ P(x) \wedge (\neg Q(x))$

2.2. Other derived rules

Derived rule. Renaming.

$$\forall x \ P(x) \vdash \forall y \ P(y) \quad \text{and} \quad \exists x \ P(x) \vdash \exists y \ P(y)$$

The variable y must not occur as a free variable in the formula $\forall x \ P(x)$ or in $\exists x \ P(x)$.

Derived rule. Switching two $\forall$ quantifiers or two $\exists$ quantifiers.

$$\forall x \ \forall y \ P(x, y) \vdash \forall y \ \forall x \ P(x, y)$$

$$\exists x \ \exists y \ P(x, y) \vdash \exists y \ \exists x \ P(x, y)$$

Derived rule. Switching $\exists \forall$ to $\forall \exists$.

$$\exists x \ \forall y \ P(x, y) \vdash \forall y \ \exists x \ P(x, y)$$

We proved it in an earlier example.

Derived rule. Interchanging $\forall$ and $\wedge$.

$$\forall x \ (P(x) \wedge Q(x)) \vdash (\forall x \ P(x)) \wedge (\forall x \ Q(x))$$

$$(\forall x \ P(x)) \wedge (\forall x \ Q(x)) \vdash \forall x \ (P(x) \wedge Q(x))$$

Thus, we can go in either direction. The formulas are interderivable.

Derived rule. Interchanging $\exists$ and $\vee$.

$$\exists x \ (P(x) \vee Q(x)) \vdash (\exists x \ P(x)) \vee (\exists x \ Q(x))$$

$$(\exists x \ P(x)) \vee (\exists x \ Q(x)) \vdash \exists x \ (P(x) \vee Q(x))$$

2.3. Proofs of the derived rules

Let us look at a few proofs of the derived rules.

Example.

The renaming rules are natural. The proofs are simple notational manipulations.

$$\forall x \ P(x) \vdash \forall y \ P(y)$$

	$\forall x \mathcal{P}(x)$	
	$\mathcal{P}(c)$	$\forall$ elimination, c arbitrary
$\vdash$	$\forall y \mathcal{P}(y)$	$\forall$ introduction
$\exists x \mathcal{P}(x) \vdash \exists y \mathcal{P}(y)$		
	$\exists x \mathcal{P}(x)$	
	$\mathcal{P}(c)$	c a constant
	$\exists y \mathcal{P}(y)$	$\exists$ introduction
$\vdash$	$\exists y \mathcal{P}(y)$	Elimination of the initial $\exists$

Example.

$\forall x \forall y \mathcal{P}(x, y) \vdash \forall y \forall x \mathcal{P}(x, y)$		
	$\forall x \forall y \mathcal{P}(x, y)$	
	$\forall y \mathcal{P}(a, y)$	Elimination of the first $\forall$, a arbitrary
	$\mathcal{P}(a, b)$	Elimination of the second $\forall$, b arbitrary
	$\forall x \mathcal{P}(x, b)$	Introduction of a $\forall$
$\vdash$	$\forall y \forall x \mathcal{P}(x, y)$	Introduction of another $\forall$

Example.

$$(\forall x \mathcal{P}(x)) \wedge (\forall x \mathcal{Q}(x)) \vdash \forall x (\mathcal{P}(x) \wedge \mathcal{Q}(x))$$

Here is the proof; it is up to you to state the rules used explicitly.

	$(\forall x \mathcal{P}(x)) \wedge (\forall x \mathcal{Q}(x))$	
	$\forall x \mathcal{P}(x)$	
	$\mathcal{P}(c)$	
	$\forall x \mathcal{Q}(x)$	
	$\mathcal{Q}(c)$	
	$\mathcal{P}(c) \wedge \mathcal{Q}(c)$	
$\vdash$	$\forall x (\mathcal{P}(x) \wedge \mathcal{Q}(x))$	

Example.

$$\neg(\forall x \mathcal{P}(x)) \vdash \exists x (\neg\mathcal{P}(x))$$

1	$\neg(\forall x \mathcal{P}(x))$	
2	$\neg(\exists x \neg\mathcal{P}(x))$	Proof by contradiction
3	$\neg\mathcal{P}(c)$	Assumption, with c arbitrary
4	$\exists x \neg\mathcal{P}(x)$	$\exists$ introduction
5	$\perp$	Contradiction between 2 and 4
6	$\mathcal{P}(c)$	From 2–5
7	$\forall x \mathcal{P}(x)$	$\forall$ introduction
8	$\perp$	Contradiction between 1 and 7
	$\exists x \neg\mathcal{P}(x)$	By contradiction from 2–8
	$\vdash$	

3. Details of the rules

We revisit the rules one by one with more explanations and examples.

3.1. Rule I. $\forall$ elimination

$\forall x \mathcal{P}(x) \vdash \mathcal{P}(c)$
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Conditions.

- If c is a constant in the language, there is no condition provided that the substitution of c for x is valid.
- If c is a term, none of its variables may become bound (captured) by a quantifier in $\mathcal{P}$ during the substitution.

Other formulations.

$$\forall x \mathcal{P}(x) \vdash \mathcal{P}[c/x]$$

We also sometimes find the slightly abusive notation $\forall x \mathcal{P}(x) \vdash \mathcal{P}(x)$.

And here is the natural deduction schema:

$$\vdash \frac{\forall x \mathcal{P}(x)}{\mathcal{P}(c)}$$

Examples.

- $\forall x \mathcal{P}(x) \wedge Q(x, c) \vdash \mathcal{P}(c) \wedge Q(c, c)$ is a correct proof. We replaced the occurrences of x by a constant c , even though this constant already appeared in the formula.

- $\forall x P(x) \vdash P(f(a, b))$ is correct if f is a function and a, b are two constants.
- $\forall x P(x) \vdash P(y)$ is also correct, provided that y is not captured by a quantifier in $\mathcal{P}$.

Counterexamples.

- $(\forall x P(x)) \rightarrow Q(a, b)$ does not prove $P(c) \rightarrow Q(a, b)$, because the “for every” is not the main connective of the sentence. On the other hand, $\forall x (P(x) \rightarrow Q(a, b))$ would indeed prove $P(c) \rightarrow Q(a, b)$.
- $\forall x \exists y Q(x, y)$ does not prove $\exists y Q(y, y)$. Indeed, in the formula “ $\exists y Q(x, y)$ ”, the variable x occurs as a free variable. Substituting y for x is therefore forbidden because the variable y inserted during the replacement would be captured by the formula’s quantifier $\exists y$.

Remark. How can we speak of a constant c if the language $\mathcal{L}$ has no constant with that name, or has no constants at all? We allow ourselves to extend the language $\mathcal{L}$ to a language $\mathcal{L}' = \mathcal{L} \cup \{c\}$, where c is a new constant symbol.

3.2. Rule II. $\forall$ introduction

$$\boxed{\mathcal{P}(c) \vdash \forall x \mathcal{P}(x)}$$

Conditions. The delicate point in using it is that c must be an arbitrary constant, that is, a constant whose symbol does not occur in any assumption currently in use.

Other formulations.

$$\begin{array}{c} \mathcal{P}(x) \vdash \forall x \mathcal{P}(x) \\ \vdash \begin{array}{c|c} \mathcal{P}(c) \\ \hline \forall x \mathcal{P}(x) \end{array} \end{array}$$

Let Γ be a set of formulas and c a constant that does not occur in the formulas of Γ .

$$\text{If } \Gamma \vdash \mathcal{P}(c) \text{ then } \Gamma \vdash \forall x \mathcal{P}(x)$$

Examples.

- $c > 0 \vdash \forall x x > 0$, provided that no assumption concerns c .
- $f(a, b) \neq 0 \vdash \forall x \forall y f(x, y) \neq 0$, provided that no assumption concerns either a or b .

Counterexamples. Many things are forbidden, and it is easy to make a mistake when applying this rule.

- $c > 0, P(c)$ does not prove $\forall x P(x)$ because the constraint $c > 0$ means that c is not an arbitrary constant. On the other hand, if we had a proof of $P(c)$ from the temporary assumption $c > 0$ (with c otherwise arbitrary), we could prove $\forall x (x > 0 \rightarrow P(x))$.
- Here is almost the same example in natural deduction notation, along with another example:

$\vdash \begin{array}{c c} c > 0 \\ \hline P(c) \\ \hline \cancel{\forall x P(x)} \end{array}$	$\vdash \begin{array}{c c} \forall x Q(c, x) \\ \hline Q(c, c) \\ \hline \cancel{\forall x Q(x, x)} \end{array}$	Valid $\forall$ elimination Invalid $\forall$ introduction
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- Replacing only some occurrences of c is forbidden. For example, $c = c$ does not prove $\forall x x = c$.

Remark. It is best if the symbol c used in the rule has not appeared at all earlier in the proof. But it is not always that simple. Here is a valid example:

	$\vdots$	c does not appear earlier
	$\left \begin{array}{c} P(c) \\ \hline Q(c) \end{array} \right.$	Temporary assumption, condition on c
	$P(c) \rightarrow Q(c)$	Conclusion in this case
	$P(c) \rightarrow Q(c)$	No assumption currently in use contains c
$\vdash$	$\forall x (P(x) \rightarrow Q(x))$	$\forall$ introduction

3.3. Rule III. $\exists$ introduction

$$\boxed{P(c) \vdash \exists x P(x)}$$

Conditions. c is a constant or a term with no free variable.

Other formulations.

$$\begin{array}{c} P[c/x] \vdash \exists x P(x) \\ \vdash \left| \begin{array}{c} P(c) \\ \hline \exists x P(x) \end{array} \right. \end{array}$$

Examples.

- $c^2 \neq c \vdash \exists x x^2 \neq x$ (seen above).
- $c^2 \neq c \vdash \exists x x^2 \neq c$ (partial replacement allowed)
- $c^2 \neq c \vdash \exists x c^2 \neq x$ (partial replacement)
- $a \times b = 12 \vdash \exists x \exists y x \times y = 12$ (two applications of the rule)

Remark. There is no danger with this rule. However, care must be taken to introduce the “there exists” at the beginning of the resulting formula. For example, the following is correct:

$$\forall x Q(x, b) \vdash \exists y \forall x Q(x, y)$$

But the same premise would not prove $\forall x \exists y Q(x, y)$.

Nonempty domain. In first-order logic, we will always assume that we are working with a nonempty domain.

3.4. Rule IV. $\exists$ elimination

$$\boxed{\exists x P(x), [P(c) \vdash Q] \vdash Q}$$

The notation in brackets corresponds to a subproof.

Conditions. c must be a **fresh constant**: first, it must not occur in any assumption currently in use; second, it must not occur in the formula Q (nor, of course, in $P(x)$). It is best if the constant occurs neither earlier in the proof nor in Q .

Other formulations.

$$\begin{aligned} \exists x \, P(x), [P[c/x] \vdash Q] &\vdash Q \\ \exists x \, P(x), [P(x) \vdash Q] &\vdash Q \end{aligned}$$

Let Γ be a set of formulas and c a constant of the language that occurs neither in Q nor in the formulas of Γ :

$$\text{If } \Gamma \vdash \exists x \, P(x) \text{ and } P(c) \vdash Q, \text{ then } \Gamma \vdash Q$$

The natural deduction notation is the clearest.

$$\begin{array}{c|c|c} & \exists x \, P(x) & \\ \hline & \begin{array}{c|c} P(c) & \text{Temporary assumption} \\ \hline \vdots \\ Q \end{array} & \\ \hline \vdash & Q & \end{array}$$

Example. Let us see how to use this rule in practice:

$$\exists x \, f(x) = 0, \forall x \, (f(x) = 0 \rightarrow g(x) = 1) \vdash \exists x \, g(x) = 1$$

This is not a direct application of the rule, but it is close:

$$\begin{array}{c|c|c} \exists x \, f(x) = 0 & & \\ \forall x \, (f(x) = 0 \rightarrow g(x) = 1) & & \\ \hline \begin{array}{c|c} f(c) = 0 & \text{Temporary assumption} \\ \hline f(c) = 0 \rightarrow g(c) = 1 & \forall \text{ elimination} \\ g(c) = 1 & \text{By implication} \\ \exists x \, g(x) = 1 & \text{This is the formula } Q \text{ in the rule} \end{array} & & \\ \hline \vdash \exists x \, g(x) = 1 & \exists \text{ elimination} & \end{array}$$

Remark. If we have several successive “there exists” statements, we must take a new fresh constant for each one. For example, if we have:

$$\exists x \, P(x), \exists y \, Q(y)$$

then in a proof we cannot choose $P(c)$ and $Q(c)$ with the same c . We must use two fresh constants c and d in order to write $P(c)$ and $Q(d)$ in the temporary assumptions.

3.5. Rule V. Equality

$$\boxed{t = s, P(t) \vdash P(s)}$$

Conditions. t and s are terms in the language. The substitution of t and s for x must be valid. There are no other conditions.

Other formulations.

$$x = y, \mathcal{P}(x) \vdash \mathcal{P}(y)$$

$$\vdash \frac{\begin{array}{c} t = s \\ \mathcal{P}(t) \end{array}}{\mathcal{P}(s)}$$

Example. $t = x + 1, t^2 \neq 0 \vdash (x + 1)^2 \neq 0$.

Remark. This rule can be replaced by the following axioms for the equality symbol:

- $\forall x \ x = x$
- $\forall x \ \forall y \ ((x = y) \rightarrow (\mathcal{P}(x) \leftrightarrow \mathcal{P}(y)))$

Exercises

Proofs in first-order logic

Exercise 1. Write proofs of the following theorems. P, Q, R denote predicates, and f, g denote functions.

1. $\forall x \ P(x), \ P(a) \rightarrow Q(a) \vdash Q(a)$
2. $\forall x \ P(x) \wedge Q(x) \vdash \exists x \ P(x)$
3. $\forall x \ \forall y \ R(x, y) \vdash \forall x \ R(x, x)$
4. $\exists x \ f(x) = g(x) \vdash \exists x \ \exists y \ f(x) = g(y)$

Exercise 2. Write proofs of the following theorems. P, Q, R denote predicates, and f denotes a function.

1. $\vdash \forall x \ \forall y \ (f(x) \neq f(y) \rightarrow x \neq y)$
2. $\forall x \ P(x), \ \forall y \ Q(y) \vdash \exists z \ P(z) \wedge Q(z)$
3. $\forall x \ P(x) \rightarrow Q(x), \ \forall y \ Q(y) \rightarrow R(y) \vdash \forall z \ P(z) \rightarrow R(z)$
4. $\forall x \ (P(x) \rightarrow Q), \ \exists x \ P(x) \vdash Q$, where Q does not depend on x .

Details of the rules

Exercise 3. Explain the error made in applying the $\forall$ elimination rule.

1. $\exists x \ \forall y \ Q(x, y)$ directly proves $\exists x \ Q(x, x)$. Bonus: give a correct proof.
2. $\forall x \ (P(x) \rightarrow Q(c))$ proves $P(c)$.
3. $\forall x \ \exists y \ Q(x, y)$ proves $\exists y \ Q(y, y)$.

Exercise 4. Explain the error made in applying the $\forall$ introduction rule.

1. $c = 1, \ c > 0$ proves $\forall x \ x > 0$.
2. $\exists x \ P(x)$ proves $\forall x \ P(x)$.
3. $(x \neq 0) \rightarrow (x^2 \neq 0)$ proves $\forall x \ x^2 \neq 0$.
4. $(x = 0) \vee (x \neq 0)$ proves $(\forall x \ x = 0) \vee (\forall x \ x \neq 0)$.
5. $f(c) = g(c)$ proves $\forall x \ f(x) = g(c)$.

Exercise 5. Explain the error made in applying the $\exists$ introduction rule.

1. $\forall x P(x)$ directly proves $\exists x P(x)$.
2. $P(a) \wedge Q(a)$ proves $\exists x P(x) \wedge Q(y)$.
3. $c = 0$, $\exists x P(x)$ proves $\exists x (x = 0) \wedge P(x)$.
4. $\forall x Q(y, y)$ proves $\exists x \forall y Q(x, y)$.
5. $\forall x Q(y, y)$ proves $\forall y \exists x Q(x, y)$.

Exercise 6. Explain the error made in applying the $\exists$ elimination rule.

1. $\exists x P(x)$, $P(0) \rightarrow Q$ proves Q .
2. $\exists x P(x)$ proves $P(c)$.
3. $\exists x (P(x) \rightarrow Q)$, $P(c)$ proves Q .
4. $\exists x P(x)$, $\exists y Q(y)$ proves $\exists z P(z) \wedge Q(z)$.

Negation of quantifiers and derived rules

Exercise 7. Write what the following formulas prove. The formula proved must no longer contain any negation to the left of a quantifier.

1. $\neg(\forall x f(x) = g(x))$
2. $\neg(\exists x P(x) \leftrightarrow Q(x))$
3. $\neg[\forall x (P(x) \rightarrow \exists y Q(y))]$

Exercise 8. Prove all the derived rules using the basic rules.