

Which Connectives Are Useful?

The connectives $\neg$ (NOT), $\wedge$ (AND), and $\vee$ (OR) are sufficient to construct every formula in true/false logic.

1. Evaluating formulas

1.1. Review

Recall that a *formula* is a finite word constructed from:

- a set $\text{Var} = \{P_1, P_2, \dots, P_n\}$ of propositional variables, where $n \geq 1$,
- parentheses (and),
- and a set Conn of logical connectives.

So far, our choice of logical connectives has been:

$$\text{Conn} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$$

Is this a sensible choice? Is it sufficient to express complex phenomena? To answer these questions, we will model the evaluation of a formula as the evaluation of a function whose inputs and output take only the values **0**/**1**.

Thus, in this chapter, the Boolean values are $\mathbb{B} = \{0, 1\}$, with **1** for T and **0** for F.

1.2. Abstract functions from $\{0, 1\}^n$ to $\{0, 1\}$

Let $n \geq 1$ be a fixed integer. An *abstract function* is a function

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

Let $x = (\varepsilon_1, \dots, \varepsilon_n) \in \{0, 1\}^n$. It is a vector whose coordinates are all **0** or **1**. In addition, $f(x)$ is either **0** or **1**.

Example.

Let $n = 3$. Define a function $f : \{0, 1\}^3 \rightarrow \{0, 1\}$ by its $2^n = 8$ values.

x	$f(x)$
$(0, 0, 0)$	0
$(0, 0, 1)$	0
$(0, 1, 0)$	1
$(0, 1, 1)$	1
$(1, 0, 0)$	0
$(1, 0, 1)$	1
$(1, 1, 0)$	0
$(1, 1, 1)$	1

Review. Let E and F be two finite sets. The total number of distinct functions $f : E \rightarrow F$ is $(\text{Card } F)^{\text{Card } E}$. Here $\text{Card } E = \text{Card}\{0, 1\}^n = 2^n$ and $\text{Card } F = \text{Card}\{0, 1\} = 2$. Thus the total number of abstract functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is $2^{(2^n)}$.

1.3. Functions from a formula

We will now construct a function $f_{\mathcal{P}} : \{0, 1\}^n \rightarrow \{0, 1\}$ from a formula.

Let $\mathcal{P}$ be a formula expressed in terms of the n propositional variables $P_1, \dots, P_n$. The **evaluation** of $\mathcal{P}$ at $x = (\varepsilon_1, \dots, \varepsilon_n) \in \{0, 1\}^n$ consists of assigning the value ε_1 (0 or 1) to P_1 , the value ε_2 to P_2 , and so on. The result of this evaluation is a value 0 or 1.

Note: A rigorous construction of this evaluation function will be given in the chapter “Compactness Theorem.”

The evaluation of $\mathcal{P}$ therefore corresponds to a function:

$$\begin{aligned} f_{\mathcal{P}} : \{0, 1\}^n &\longrightarrow \{0, 1\} \\ x = (\varepsilon_1, \dots, \varepsilon_n) &\longmapsto f_{\mathcal{P}}(\varepsilon_1, \dots, \varepsilon_n) \end{aligned}$$

This is another way of presenting the truth table of the formula $\mathcal{P}$.

Example.

Let $\mathcal{P}$ be the following formula:

$$\mathcal{P} : (P_1 \vee P_2) \wedge (\neg P_1 \vee P_3)$$

We obtain a function $f_{\mathcal{P}} : \{0, 1\}^3 \rightarrow \{0, 1\}$, which we compute one value at a time. For example, at $x = (0, 1, 0)$ we assign (F, T, F) to the variables (P_1, P_2, P_3) . The formula $\mathcal{P}$ then evaluates to $(F \vee T) \wedge (\neg F \vee F)$, and hence to T. Thus $f_{\mathcal{P}}(0, 1, 0) = 1$.

x	$f_{\mathcal{P}}(x)$
$(0, 0, 0)$	0
$(0, 0, 1)$	0
$(0, 1, 0)$	1
$(0, 1, 1)$	1
$(1, 0, 0)$	0
$(1, 0, 1)$	1
$(1, 1, 0)$	0
$(1, 1, 1)$	1

Here we observe that our function $f_{\mathcal{P}}$ is exactly the same as the abstract function f defined in the

previous example.

1.4. Can every function be represented?

In the previous example, we had a formula $\mathcal{P}$ whose associated function $f_{\mathcal{P}}$ was the same as an abstract function defined earlier. But is this always possible? Given an abstract function $f : \{0, 1\}^n \rightarrow \{0, 1\}$, can we find a formula $\mathcal{P}$ such that $f_{\mathcal{P}}$ is exactly the function f ?

If the answer were no, it would mean that our connectives are not sufficient to model every function and that we would have to invent new connectives to represent some functions. The question is all the more pressing because, when n is large, the number of functions f to represent is 2^{2^n} , which is very large indeed.

1.5. Unary connectives

Before turning to a general theoretical result, let us begin by studying the cases $n = 1$ and $n = 2$ by hand.

For $n = 1$, there are $4 = 2^{2^1}$ functions $f : \{0, 1\} \rightarrow \{0, 1\}$. Here are the 4 possible truth tables:

	$\top$		$\perp$		id		$\neg$
0	1	0	0	0	0	0	1
1	1	1	0	1	1	1	0

These 4 functions are $\top$ (always true), $\perp$ (always false), id (identity), and $\neg$ (negation).

The case $n = 1$ is summarized in a single table.

Symbol	Representation	Name
$\top$	$P \vee \neg P$	tautology / always true
$\perp$	$P \wedge \neg P$	contradiction / always false
id	P	identity
$\neg$	$\neg P$	negation

The second column gives a representation by a formula using the connectives $\neg$, $\wedge$, and $\vee$.

1.6. Binary connectives

For $n = 2$, there are $16 = 2^{2^2}$ functions $f : \{0, 1\}^2 \rightarrow \{0, 1\}$. Below is a list of all these functions and their representations by formulas. The second column gives a representation by a formula using the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. The last column gives the 4 values of the function when (P, Q) is successively $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$.

Symbol	Representation	Name	Values
$\perp$	$P \wedge \neg P$	always false / contradiction	0000
$\wedge$	$P \wedge Q$	AND/ conjunction	0001
	$P \wedge \neg Q$	P and (not Q)	0010
	P	left projection	0011
	$\neg P \wedge Q$	(not P) and Q	0100
	Q	right projection	0101
$\oplus$	$(P \vee Q) \wedge \neg(P \wedge Q)$	XOR / exclusive or	0110
$\vee$	$P \vee Q$	OR/ disjunction	0111
$\downarrow$	$\neg(P \vee Q)$	NOR	1000
$\leftrightarrow$	$P \leftrightarrow Q$	equivalence	1001
	$\neg Q$	negation of Q	1010
	$Q \rightarrow P$	Q implies P	1011
	$\neg P$	negation of P	1100
$\rightarrow$	$P \rightarrow Q$	P implies Q	1101
$\uparrow$	$\neg(P \wedge Q)$	NAND / Sheffer	1110
$\top$	$P \vee \neg P$	always true / tautology	1111

For example, the connective $\downarrow$ /NOR satisfies $0 \downarrow 0 = 1$, $0 \downarrow 1 = 0$, $1 \downarrow 0 = 0$, $1 \downarrow 1 = 0$. But $P \downarrow Q$ is in fact represented by the formula $\neg(P \vee Q)$. Thus we do not really need a special symbol for this operation. However, two new connectives are useful in practice: XOR and NAND.

- The **exclusive or**, denoted by $\oplus$ /XOR, is true when one of the terms is true, but not both. It corresponds to addition $x + y$ modulo 2.
- The connective $\uparrow$ /NAND, also called the Sheffer stroke, is represented as the negation of AND:
 $P \uparrow Q = \neg(P \wedge Q)$.

2. A complete set of connectives

2.1. Theorem

Theorem 1.

The set $\{\neg, \wedge\}$ consisting of the connectives NOT and AND forms a complete set of connectives.

We will see the proof in the next section. This means that, for every $n \geq 1$ and every $f : \{0, 1\}^n \rightarrow \{0, 1\}$, there is a formula $\mathcal{P}$ involving only the connectives $\neg$ and $\wedge$ such that the function $f_{\mathcal{P}}$ arising from $\mathcal{P}$ is exactly f . In other words, f and $f_{\mathcal{P}}$ have the same truth table.

The important point to remember is that a finite number of connectives can represent every conceivable Boolean function, whatever n may be.

Another way to understand this theorem is to say that, for any formula (with any connectives), we can find a logically equivalent formula containing only the connectives $\neg$ and $\wedge$. Recall that two formulas $\mathcal{P}$ and $\mathcal{Q}$ are logically equivalent (written $\mathcal{P} \equiv \mathcal{Q}$) if they have the same truth table.

Not such a silly question! If we need only $\{\neg, \wedge\}$ to represent all our functions, why do we add the other connectives $\vee$, $\rightarrow$, and $\leftrightarrow$? Quite simply for practical reasons: to make formulas shorter and, above all, to make their logical meaning easier to understand.

2.2. Examples

1. For $n = 1$ and $n = 2$, we have already seen how to represent every function (allowing ourselves to use $\vee$, $\rightarrow$, and $\leftrightarrow$ as well).
2. The same function can be represented by different formulas. For example, $P \wedge \neg Q$, $\neg Q \wedge P \wedge \neg Q$, and $(\neg \neg P) \wedge (\neg Q)$ represent the same function. The representation is even less unique if all the connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ are allowed. For example, $P \rightarrow Q$, $\neg P \vee Q$, and $\neg(P \wedge \neg Q)$ are formulas that represent the same function.
3. Let $n = 3$, and let f be the function that equals **1** at $(0, 0, 0)$ and $(1, 1, 1)$ and equals **0** everywhere else. This function is represented by the formula

$$\mathcal{P} : \quad (\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge R)$$

Problem: the formula contains a $\vee$ (OR), whereas we would like to use only $\neg$ and $\wedge$. We will see the solution in the next subsection.

2.3. The usual connectives from $\neg$ and $\wedge$

Since the connectives $\neg$ and $\wedge$ are sufficient, they must allow us to recover our other favorite connectives: $\vee$, $\rightarrow$, and $\leftrightarrow$.

Indeed:

$$\begin{aligned} P \vee Q &\equiv \neg(\neg P \wedge \neg Q) \\ P \rightarrow Q &\equiv \neg P \vee Q \equiv \neg(P \wedge \neg Q) \\ P \leftrightarrow Q &\equiv (P \rightarrow Q) \wedge (Q \rightarrow P) \equiv (\neg(P \wedge \neg Q)) \wedge (\neg(Q \wedge \neg P)) \end{aligned}$$

Recall that the notation $\equiv$ means that the formulas have exactly the same truth tables. Of course, these relations also hold for formulas $\mathcal{P}$ and $\mathcal{Q}$ (in place of the variables P and Q).

Thus, every formula expressed using the connectives $\mathcal{C} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ can be rewritten as a logically equivalent formula containing only the connectives $\mathcal{C}_0 = \{\neg, \wedge\}$.

2.4. Other complete sets of connectives

The set $\mathcal{C}_1 = \{\neg, \rightarrow\}$ is also a complete set of connectives. The proof follows easily from the theorem: it is enough to show how to generate the connective $\wedge$ from $\neg$ and $\rightarrow$:

$$P \wedge Q \equiv \neg(P \rightarrow \neg Q)$$

Thus, given any formula, the theorem gives us a logically equivalent formula using only $\neg$ and $\wedge$. We can then replace each $\wedge$ using the logical equivalence above to obtain a formula containing only $\neg$ and $\rightarrow$.

Even more remarkably, a single connective can generate all of them! $\mathcal{C}_2 = \{\uparrow\}$ is a complete set. Recall that $\uparrow$ / NAND is the connective corresponding to the negation of AND:

$$P \uparrow Q \equiv \neg(P \wedge Q)$$

Suppose that the connective $\uparrow$ is defined. We can then recover $\neg$ and $\wedge$:

$$\neg P \equiv P \uparrow P \quad P \wedge Q \equiv \neg(P \uparrow Q) \equiv (P \uparrow Q) \uparrow (P \uparrow Q)$$

Since $\uparrow$ can be used to reconstruct both $\neg$ and $\wedge$, and $\{\neg, \wedge\}$ is complete, it follows that $\{\uparrow\}$ is a complete set.

From a computer science perspective, this means that a single type of logic gate, the NAND gate, can model any Boolean function.

3. Proof

This section is devoted to the proof of the theorem. We will prove that $\mathcal{C} = \{\neg, \wedge, \vee\}$ is a complete set of connectives. Since we have seen that $\vee$ is generated by $\{\neg, \wedge\}$ through $P \vee Q \equiv \neg(\neg P \wedge \neg Q)$, this will prove that $\mathcal{C}_0 = \{\neg, \wedge\}$ is a complete set of connectives.

We will represent each abstract function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ by an explicit formula called a **disjunctive normal form**.

3.1. Step 1

We begin by representing some very simple functions.

Fix $a = (\varepsilon_1, \dots, \varepsilon_n) \in \{0, 1\}^n$. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be the function such that $f(a) = 1$ and $f(x) = 0$ if $x \neq a$.

The following formula $\mathcal{P}_a$ represents this function:

$$\mathcal{P}_a : \quad \varepsilon_1 P_1 \wedge \varepsilon_2 P_2 \wedge \dots \wedge \varepsilon_n P_n$$

Here, P_i denotes the i th propositional variable, that is, the i th coordinate of $\{0, 1\}^n$. Recall that $\varepsilon_i = 0$ or $\varepsilon_i = 1$, and define

$$\varepsilon_i P_i = \begin{cases} P_i & \text{if } \varepsilon_i = 1 \\ \neg P_i & \text{if } \varepsilon_i = 0 \end{cases}$$

We will use the following notation:

$$\mathcal{P}_a : \quad \bigwedge_{i=1}^n \varepsilon_i P_i$$

This notation is analogous to $\sum_{i=1}^n x_i$ for $x_1 + x_2 + \dots + x_n$, or to $\prod_{i=1}^n x_i$ for $x_1 x_2 \dots x_n$.

For example, the formula $P \wedge \neg Q \wedge \neg R$ gives a function that equals 1 at $(1, 0, 0)$ and equals 0 everywhere else.

Recall that $a = (\varepsilon_1, \dots, \varepsilon_n)$. Let us prove that $\mathcal{P}_a$ represents the function described above:

$$\begin{aligned} \mathcal{P}_a \text{ is true} & \quad \text{iff} \quad \varepsilon_i P_i \text{ is true, for every } i = 1, \dots, n \\ & \quad \text{iff} \quad \begin{cases} P_i \text{ is true} & \text{if } \varepsilon_i = 1 \\ P_i \text{ is false} & \text{if } \varepsilon_i = 0 \end{cases} \quad \text{for every } i = 1, \dots, n \end{aligned}$$

Therefore the associated function $f_{\mathcal{P}_a}$ satisfies $f_{\mathcal{P}_a}(x_1, \dots, x_n) = 1$ if and only if $x_i = \varepsilon_i$ for every $i = 1, \dots, n$. In other words, $f_{\mathcal{P}_a}(x) = 1$ if and only if $x = a$.

3.2. Step 2

Let $a_1, a_2, \dots, a_k \in \{0, 1\}^n$ be a list of k elements of $\{0, 1\}^n$ (with $k \geq 1$).

Let $\mathcal{P}_{a_1, \dots, a_k}$ be the following formula:

$$\mathcal{P}_{a_1, \dots, a_k} : \quad \mathcal{P}_{a_1} \vee \mathcal{P}_{a_2} \vee \dots \vee \mathcal{P}_{a_k}$$

In other words:

$$\mathcal{P}_{a_1, \dots, a_k} : \quad \bigvee_{j=1}^k \mathcal{P}_{a_j} = \bigvee_{j=1}^k \bigwedge_{i=1}^n \varepsilon_{i,j} P_i$$

where we have written $a_j = (\varepsilon_{1,j}, \varepsilon_{2,j}, \dots, \varepsilon_{n,j})$.

Let $f_{a_1, \dots, a_k}$ denote the function associated with the formula $\mathcal{P}_{a_1, \dots, a_k}$. Then:

$$f_{a_1, \dots, a_k}(x) = 1 \quad \text{iff} \quad x \in \{a_1, a_2, \dots, a_k\}$$

Example. For the formula $\mathcal{P}_a \vee \mathcal{P}_b$, the associated function $f_{a,b}$ satisfies $f(a) = 1$, $f(b) = 1$, and $f(x) = 0$ for every other x .

Let us prove our claim.

- If $x \in \{a_1, a_2, \dots, a_k\}$, then one of the $\mathcal{P}_{a_j}$ is true, so $\mathcal{P}_{a_1, \dots, a_k}$ is true. Thus $f_{a_1, \dots, a_k}(x) = 1$.
- If $x \notin \{a_1, a_2, \dots, a_k\}$, then all the $\mathcal{P}_{a_1}, \dots, \mathcal{P}_{a_k}$ are false, so $\mathcal{P}_{a_1, \dots, a_k}$ is false. Thus $f_{a_1, \dots, a_k}(x) = 0$.

3.3. Step 3

Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be any abstract function.

- If $f(x) = 0$ for every $x \in \{0, 1\}^n$, set $\mathcal{P} = P_1 \wedge \neg P_1$ (always false); then $f_{\mathcal{P}} = f$.
- Otherwise, let $\{a_1, \dots, a_k\}$ be the elements of $\{0, 1\}^n$ for which $f(x) = 1$. Then $f = f_{a_1, \dots, a_k}$ by Step 2.

This completes the proof.

3.4. Remarks

- The decomposition

$$\mathcal{P} : \bigvee_{j=1}^k \bigwedge_{i=1}^n \varepsilon_{i,j} P_i$$

is called the *disjunctive normal form*.

This normal form is useful because the decomposition is unique up to order (of course, $P \vee Q \equiv Q \vee P$ and $P \wedge Q \equiv Q \wedge P$).

- A variant is the *conjunctive normal form*:

$$\mathcal{Q} : \bigwedge_{j=1}^k \bigvee_{i=1}^n \delta_{i,j} P_i$$

where $\delta_{i,j} \in \{0, 1\}$. Its expression is also unique (up to order).

Example.

Let $\mathcal{P}$ be the formula $P \leftrightarrow Q$. Its disjunctive normal form is:

$$\mathcal{P} \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

Its conjunctive normal form is:

$$\mathcal{P} \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$$

Exercises

Evaluating formulas

Exercise 1. Find a formula for $P \oplus Q$ (exclusive or) involving only the connectives $\neg$, $\wedge$, and $\vee$. Then find a formula involving only $\neg$ and $\leftrightarrow$.

Exercise 2.

1. For $n = 3$, how many functions $f : \{0, 1\}^3 \rightarrow \{0, 1\}$ are there?
2. The “majority” function returns **1** if at least two of the variables P, Q, R equal **1**. Find a formula for this function involving only the connectives $\neg$, $\wedge$, and $\vee$.
3. When is $P \oplus Q \oplus R$ true? What is its relationship with $P \leftrightarrow (Q \leftrightarrow R)$?
4. The “exactly one” function returns **1** when exactly one of the variables P, Q , or R equals **1**. Find a formula for this function involving only the connectives $\neg$, $\wedge$, and $\vee$.

Exercise 3. Let $n \geq 2$.

1. Find a formula that returns true unless all the variables $P_1, \dots, P_n$ are true.
2. What formula returns true when at least two variables are true?

A complete set of connectives

Exercise 4.

1. Show that $\mathcal{C} = \{\neg, \leftrightarrow\}$ is a complete set of connectives.
2. Show that $\mathcal{C} = \{\vee, \leftrightarrow, \perp\}$ is a complete set of connectives. Recall that $\perp$ is the always-false formula.
3. Show that $\mathcal{C} = \{\wedge, \vee\}$ is not a complete set of connectives.
4. Show that $\mathcal{C} = \{\wedge, \oplus\}$ is not a complete set of connectives ($\oplus$ denotes exclusive or).

Exercise 5. Show that $\{\downarrow\}$ forms a complete set. Recall that $P \downarrow Q \equiv \neg(P \vee Q)$.

Proof

Exercise 6. Let $n = 2$. Only the connectives $\neg, \wedge, \vee$ are allowed.

1. Find a formula that represents the function $f : \{0, 1\}^2 \rightarrow \{0, 1\}$, which equals 1 at $(1, 0)$ and equals 0 everywhere else.
2. Answer the same question for $g : \{0, 1\}^2 \rightarrow \{0, 1\}$, which equals 1 at $(1, 1)$ and 0 everywhere else.
3. Deduce a formula for $h : \{0, 1\}^2 \rightarrow \{0, 1\}$, which equals 1 exactly at $(1, 0)$ and $(1, 1)$. Simplify this formula.

Exercise 7. Let $n = 2$.

1. What function is associated with the formula $P \rightarrow Q$?
2. Find the associated disjunctive normal form.

Exercise 8. Let $n = 3$. Only the connectives $\neg, \wedge, \vee$ are allowed.

1. Find a formula that represents the function f , which equals 1 only at $(1, 0, 1)$.
2. Find a formula that represents the function g , which equals 0 only at $(0, 1, 1)$.
3. Find a formula that represents the function h , which equals 1 only at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Find a disjunctive normal form and a conjunctive normal form for it.