

Completeness Theorem

The Completeness Theorem states that, in first-order logic, everything that is true is provable. We state this theorem, its variants, and its applications.

1. Completeness Theorem

1.1. Statement

Let $\mathcal{L}$ be a first-order language.

Theorem 1 (Completeness Theorem).

$$\boxed{\text{If } \mathcal{P}_1, \dots, \mathcal{P}_l \models \mathcal{Q} \text{ then } \mathcal{P}_1, \dots, \mathcal{P}_l \vdash \mathcal{Q}}$$

This can be read as follows: if $\mathcal{Q}$ is satisfied whenever the assumptions $\mathcal{P}_1, \dots, \mathcal{P}_l$ are satisfied, then there is a proof of $\mathcal{Q}$ from $\mathcal{P}_1, \dots, \mathcal{P}_l$.

Here is the variant for a set Γ of formulas, which may be countably infinite:

Theorem 2 (Completeness Theorem).

$$\text{If } \Gamma \models \mathcal{Q} \text{ then } \Gamma \vdash \mathcal{Q}.$$

The conclusion $\Gamma \vdash \mathcal{Q}$ is an existence result: “there is a proof of $\mathcal{Q}$ from the formulas in Γ ”. However, the theorem is not constructive and gives no concrete information about this proof.

The proof of the Completeness Theorem is fairly demanding. It has points in common with the Compactness Theorem for true/false logic, but with an additional level of difficulty. The proof will be given in the next chapter. We will now focus on the different restatements and applications.

1.2. Equivalence

In the previous chapter, we saw the Soundness Theorem, which is exactly the converse of the Completeness Theorem. Thus, we obtain an equivalence that is also called the Completeness Theorem:

Theorem 3 (Completeness Theorem).

$$\boxed{\Gamma \models \mathcal{Q} \text{ iff } \Gamma \vdash \mathcal{Q}}$$

Thus, in first-order logic, logical consequence and provability are equivalent notions.

1.3. What about incompleteness?

Is there not a contradiction between obtaining the Completeness Theorem and, at the end of the book, obtaining the Incompleteness Theorem, whose name seems to suggest the opposite? In fact, the Completeness Theorem applies to first-order logic, which is a fairly rudimentary logic, but it uses the quite strong assumption $\Gamma \models \mathcal{Q}$, since logical consequence requires the formula to be true in every possible structure.

The Incompleteness Theorem does not concern all structures, but rather more specific theories, typically ones that contain the natural numbers $\mathbb{N}$ together with the axioms of arithmetic (the Peano axioms). We will see that, for the arithmetic of the natural numbers, there are formulas $\mathcal{Q}$ such that neither $\mathcal{Q}$ nor $\neg\mathcal{Q}$ is provable. In other words, there are true but unprovable formulas, but all this holds only for the structure associated with the arithmetic of the natural numbers. This does not contradict the Completeness Theorem, because the assumption “ $\Gamma \models \mathcal{Q}$ ” is not verified for every structure, but only for certain specific structures; the conditions of the Completeness Theorem are therefore not met.

Thus, the Completeness and Incompleteness Theorems are both true, but they do not apply to the same situations and therefore do not contradict each other.

2. Applications to theories

We will recall some definitions and introduce new notions. Let $\mathcal{L}$ be a first-order language.

2.1. Consistent theory, model

- A **closed formula** (also called a **sentence**) is a formula with no free variables; that is, every occurrence of a variable lies within the scope of a quantifier.
- A **theory** $\mathcal{T}$ is a set of closed formulas. (Note: some authors call any set of formulas a “theory”.)
- A structure $\mathcal{S}$ is a **model** of $\mathcal{T}$ if $\mathcal{T}$ is satisfied in $\mathcal{S}$, denoted $\models_{\mathcal{S}} \mathcal{T}$; that is, the interpretation of every formula in $\mathcal{T}$ is true in $\mathcal{S}$. (Since the formulas are closed, there is no assignment to consider.)

Here is a very important new notion.

Definition.

A theory $\mathcal{T}$ is **inconsistent** if there is a formula $\mathcal{Q}$ such that:

$$\mathcal{T} \vdash \mathcal{Q} \quad \text{and} \quad \mathcal{T} \vdash \neg\mathcal{Q}.$$

Otherwise, $\mathcal{T}$ is said to be **consistent**.

Equivalently:

$\mathcal{T} \text{ is inconsistent iff } \mathcal{T} \vdash \perp$

where $\perp$ denotes a contradiction. Thus, a contradiction can never be proved from a consistent theory (which is reassuring).

2.2. Natural deduction lemma

Let Γ denote a set of formulas, and let $\mathcal{P}$ and $\mathcal{Q}$ be formulas.

Lemma 1 (Natural deduction lemma).

$\Gamma, \mathcal{P} \vdash \mathcal{Q} \quad \text{iff} \quad \Gamma \vdash (\mathcal{P} \rightarrow \mathcal{Q})$

Proof. The reverse implication $\Leftarrow$ is simply the *modus ponens* rule $\mathcal{P}, \mathcal{P} \rightarrow \mathcal{Q} \vdash \mathcal{Q}$.

The forward implication $\Rightarrow$ is immediate when we have a proof in natural deduction (hence the name of the lemma); it is simply a matter of rewriting the proof:

Proof of $\Gamma, \mathcal{P} \vdash \mathcal{Q}$

$\vdash$	Γ	
	$\mathcal{P}$	Assumptions
	$\vdots$	Steps
	$\mathcal{Q}$	Conclusion

Proof of $\Gamma \vdash (\mathcal{P} \rightarrow \mathcal{Q})$

$\vdash$	Γ	
	$\mathcal{P}$	Temporary assumption
	$\vdots$	Same steps
	$\mathcal{Q}$	Intermediate conclusion
	$\mathcal{P} \rightarrow \mathcal{Q}$	Conclusion

□

Let $\mathcal{T}$ be a theory and $\mathcal{Q}$ a closed formula.

Corollary 1.

$$\mathcal{T} \vdash \mathcal{Q} \quad \text{iff} \quad \mathcal{T} \cup \{\neg \mathcal{Q}\} \text{ is inconsistent}$$

Proof.

$$\begin{aligned} & \mathcal{T} \cup \{\neg \mathcal{Q}\} \text{ is inconsistent} \\ \text{iff } & \mathcal{T} \cup \{\neg \mathcal{Q}\} \vdash \perp \quad \text{by definition} \\ \text{iff } & \mathcal{T} \vdash (\neg \mathcal{Q} \rightarrow \perp) \quad \text{by the natural deduction lemma} \\ \text{iff } & \mathcal{T} \vdash \neg(\neg \mathcal{Q}) \quad \text{since, by definition, } \neg \mathcal{P} \text{ means } \mathcal{P} \rightarrow \perp \\ \text{iff } & \mathcal{T} \vdash \mathcal{Q} \quad \text{by double negation} \end{aligned}$$

□

2.3. Different formulations of the Completeness Theorem

We will state the Completeness Theorem along with several restatements.

Setting: $\mathcal{T}$ is a theory and $\mathcal{Q}$ is a closed formula.

Standard formulation:

Theorem 4 (Theorem A).

$$\mathcal{T} \models \mathcal{Q} \quad \text{iff} \quad \mathcal{T} \vdash \mathcal{Q}$$

Restatement:

Theorem 5 (Theorem B).

$$\mathcal{T} \text{ is consistent} \quad \text{iff} \quad \mathcal{T} \text{ has a model}$$

We could state a similar result for any set of formulas:

$$\Gamma \text{ is consistent} \quad \text{iff} \quad \Gamma \text{ is satisfiable.}$$

2.4. Proof of Theorem B from Theorem A

$$\begin{aligned}
 & \mathcal{T} \text{ is inconsistent} \\
 \text{iff } & \mathcal{T} \vdash \perp \quad \text{by definition} \\
 \text{iff } & \mathcal{T} \models \perp \quad \text{by Theorem A} \\
 \text{iff } & \mathcal{T} \text{ is satisfied by no structure } \mathcal{S} \\
 \text{iff } & \mathcal{T} \text{ has no model}
 \end{aligned}$$

In an exercise (and in the next chapter), we will see that Theorem A can be proved from Theorem B.

3. Application to finiteness

We are interested in the case where the set of assumptions is infinite (but countable). We will see that only finitely many assumptions are needed. We will obtain two results: one about proofs (the finiteness theorem) and one about satisfaction (the Compactness Theorem for first-order logic, analogous to the theorem in the chapter “Compactness Theorem” for true/false logic).

3.1. Finiteness theorem

Let Γ be a (possibly infinite) set of formulas, and let $\mathcal{Q}$ be a formula in a language $\mathcal{L}$.

Theorem 6 (Finiteness theorem).

If $\Gamma \vdash \mathcal{Q}$, then there is a finite subset $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \vdash \mathcal{Q}$.

Note: the converse is of course true.

Proof. The proof is based on the very nature of a proof. The statement $\Gamma \vdash \mathcal{Q}$ means that there is a proof of $\mathcal{Q}$ from the formulas in Γ . Consider such a proof. This proof consists of finitely many steps, and at each step a rule is applied. Each rule involves only finitely many formulas from Γ . Let Γ_0 denote the set of formulas from Γ that occur in this proof. We have therefore obtained a proof of $\mathcal{Q}$ from Γ_0 , so $\Gamma_0 \vdash \mathcal{Q}$ with Γ_0 finite. $\square$

3.2. Compactness Theorem

The corresponding result for satisfaction is the Compactness Theorem for first-order logic, similar to the one encountered for true/false logic in the chapter “Compactness Theorem” in Part One.

Theorem 7 (Compactness Theorem).

Γ is satisfiable iff every finite subset $\Gamma_0 \subset \Gamma$ is satisfiable.

Restatement: $\Gamma \models \mathcal{Q}$ iff $\Gamma_0 \models \mathcal{Q}$ for every finite subset $\Gamma_0 \subset \Gamma$.

Note: in each case, the forward implication $\Rightarrow$ is immediate.

For theories, the interesting implication can be restated as follows:

Theorem 8.

If every finite subset of a theory $\mathcal{T}$ has a model, then $\mathcal{T}$ itself has a model.

It is worth taking the time to reflect on how profound such a theorem is. Every finite subset has a model, but of course this model may be different for each Γ_0 . Nevertheless, in the end we find a model common to all the formulas.

Proof of the Compactness Theorem. We have already said that the forward implication $\Rightarrow$ is immediate. We will prove the reverse implication $\Leftarrow$ by contraposition: “if Γ is unsatisfiable, then there is a finite Γ_0 that is unsatisfiable.”

Γ is unsatisfiable
 therefore $\Gamma \models \perp$ since Γ is never satisfied anyway
 therefore $\Gamma \vdash \perp$ by the Completeness Theorem, so Γ is inconsistent
 therefore $\Gamma_0 \vdash \perp$ for some finite $\Gamma_0 \subset \Gamma$, by the finiteness theorem
 therefore $\Gamma_0 \models \perp$ by the Soundness Theorem
 therefore Γ_0 is unsatisfiable
 therefore there exists a finite, unsatisfiable Γ_0
 therefore not (every finite Γ_0 is satisfiable)

□

4. Closure theorem

The closure theorem explains how to pass from a statement of the Completeness Theorem for closed formulas to a statement for arbitrary formulas.

4.1. Statement

Theorem 9 (Theorem A).

Let $\mathcal{T}$ be a theory and $\mathcal{Q}$ a closed formula.

If $\mathcal{T} \models \mathcal{Q}$ then $\mathcal{T} \vdash \mathcal{Q}$.

Theorem 10 (Theorem C).

Now let Γ be an arbitrary set of formulas and $\mathcal{Q}$ an arbitrary formula.

If $\Gamma \models \mathcal{Q}$ then $\Gamma \vdash \mathcal{Q}$.

Theorem 11 (Closure theorem).

Theorem A implies Theorem C.

In other words, to prove the Completeness Theorem in full generality, it is enough to prove it for closed formulas.

4.2. Closure

Before proving Theorem C, let us introduce a notion of closure by adding new constants.

Definition.

Let $\mathcal{P}$ be a formula in a language $\mathcal{L}$ whose free variables are denoted by $x_1, \dots, x_n$. Let $\mathcal{L}^c$ denote the language obtained from $\mathcal{L}$ by adding n new constants $c_1, \dots, c_n$ (these constants must not occur in $\mathcal{L}$). Let $\mathcal{P}^c$ denote the **substitution of the new constants for the free variables** in $\mathcal{P}$, that is, the formula obtained from $\mathcal{P}$ by replacing each free variable x_i with the new constant c_i .

By construction, $\mathcal{P}^c$ is a closed formula (in the language $\mathcal{L}^c$).

For a set of formulas Γ and a formula $\mathcal{Q}$, let $x_1, \dots, x_n$ denote the set of free variables in $\Gamma \cup \{\mathcal{Q}\}$. In this way, we construct a language $\mathcal{L}^c$ and closed formulas Γ^c (the closure of the formulas in Γ by the new constants) and $\mathcal{Q}^c$.

Here are some useful properties for the proof, which are essentially manipulations of notation.

- Point a.

$$\Gamma \models \mathcal{Q} \quad \text{implies} \quad \Gamma^c \models \mathcal{Q}^c$$

Let us justify this in detail. Consider a structure $\mathcal{S}^c$ (for the language $\mathcal{L}^c$) that satisfies Γ^c . Let $d_i = c_i^{\mathcal{S}^c}$ denote the interpretation of the constant c_i in $\mathcal{S}^c$.

We can construct a structure $\mathcal{S}$ (for the language $\mathcal{L}$) by forgetting the new constants $c_1, \dots, c_n$, while at the same time constructing an assignment s by setting $s(x_i) = d_i$ for each free variable x_i . For a formula $\mathcal{P}$ (in Γ or $\mathcal{Q}$), we then have:

$$\models_{\mathcal{S}^c} \mathcal{P}^c \quad \text{iff} \quad \models_{\mathcal{S}, s} \mathcal{P}$$

(This follows directly from substituting the constant c_i for each free variable x_i .)

Since $\models_{\mathcal{S}^c} \Gamma^c$, we have $\models_{\mathcal{S}, s} \Gamma$ (by the preceding substitution). By assumption, $\Gamma \models \mathcal{Q}$, so $\models_{\mathcal{S}, s} \mathcal{Q}$, and hence, going back, $\models_{\mathcal{S}^c} \mathcal{Q}^c$.

- Point b. The new constants lemma:

$$\Gamma^c \vdash \mathcal{Q}^c \quad \text{implies} \quad \Gamma \vdash \mathcal{Q}$$

Indeed, suppose we have a natural-deduction proof of $\mathcal{Q}^c$ from Γ^c . Copy this proof, replacing every occurrence of the new constants c_i with the variables x_i on every line. This gives a proof of $\mathcal{Q}$ from Γ .

Proof of $\Gamma^c \vdash \mathcal{Q}^c$			Proof of $\Gamma \vdash \mathcal{Q}$		
$\vdash$	$\mathcal{P}_1^c$	Assumptions Γ^c	$\vdash$	$\mathcal{P}_1$	Assumptions Γ
	$\vdots$			$\vdots$	
	$\mathcal{P}_l^c$			$\mathcal{P}_l$	
	$\vdots$	Proof		$\vdots$	Proof
	$\dots c_i \dots$	The new constants		$\dots x_i \dots$	Back to the variables
	$\vdots$			$\vdots$	
	$\mathcal{Q}^c$	Conclusion		$\mathcal{Q}$	Conclusion

4.3. Proof of Theorem C from Theorem A

$$\begin{array}{llll}
 & \Gamma \models \mathcal{Q} & & \\
 \text{therefore} & \Gamma^c \models \mathcal{Q}^c & \text{by point a} & \\
 \text{therefore} & \Gamma^c \vdash \mathcal{Q}^c & \text{by Theorem A for closed formulas} & \\
 \text{therefore} & \Gamma \vdash \mathcal{Q} & \text{by point b} &
 \end{array}$$

4.4. Universal closure

There is another notion of closure, called *universal closure*.

Definition.

Let $\mathcal{P}$ be a formula whose free variables are denoted by $x_1, \dots, x_n$. The *universal closure* of $\mathcal{P}$ is denoted by $\overline{\mathcal{P}}$ (or also $\forall \mathcal{P}$):

$$\overline{\mathcal{P}} : \quad \forall x_1 \forall x_2 \dots \forall x_n \mathcal{P}$$

Exercises

Completeness Theorem

Exercise 1. Let Γ be a set of formulas and $\mathcal{P}, \mathcal{Q}$ formulas such that:

$$\Gamma \vdash \mathcal{P} \quad \text{and} \quad \mathcal{P} \vdash \mathcal{Q} \quad (*)$$

The goal of the exercise is to obtain $\Gamma \vdash \mathcal{Q}$.

1. *First method.* Starting from a natural-deduction proof of $\Gamma \vdash \mathcal{P}$ and a natural-deduction proof of $\mathcal{P} \vdash \mathcal{Q}$, write a proof of $\Gamma \vdash \mathcal{Q}$.
2. *Second method.*
 - (a) What statements follow from our assumptions $(*)$ by applying the Soundness Theorem?
 - (b) Under the assumptions $(*)$, show that $\Gamma \models \mathcal{Q}$.
 - (c) Conclude.

Exercise 2. Let $\mathcal{L}$ be a language with a binary predicate $R(x, y)$. Consider the following three formulas:

1. symmetry $\mathcal{S} : \forall x \forall y (R(x, y) \rightarrow R(y, x))$
2. transitivity $\mathcal{T} : \forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$
3. reflexivity $\mathcal{Q} : \forall x R(x, x)$

A relation R for which these three formulas are true is called an *equivalence relation*. We ask whether the two axioms $\mathcal{S}$ and $\mathcal{T}$ alone are sufficient; that is, do we have $\mathcal{S}, \mathcal{T} \vdash \mathcal{Q}$?

1. A student proposes the following proof:

Let x be an arbitrary element. Choose an element y such that $R(x, y)$. By axiom $\mathcal{S}$ (symmetry), we also have $R(y, x)$. Applying axiom $\mathcal{T}$ (transitivity) to $R(x, y)$ and $R(y, x)$ gives $R(x, x)$. The relation is therefore reflexive, and $\mathcal{Q}$ is proved.

What is the fundamental logical error in this reasoning?

2. Find a structure $\mathcal{S}$, that is, define a domain E and specify the relation R on it, such that the formulas $\mathcal{S}$ and $\mathcal{T}$ are true in it, but $\mathcal{Q}$ is not.
3. Do we have $\mathcal{S}, \mathcal{T} \vdash \mathcal{Q}$?

Applications to theories

Exercise 3. Show that the following two definitions of an *inconsistent* theory $\mathcal{T}$ are equivalent:

- (i) there is a $\mathcal{Q}$ such that $\mathcal{T} \vdash \mathcal{Q}$ and $\mathcal{T} \vdash \neg \mathcal{Q}$,
- (ii) $\mathcal{T} \vdash \perp$.

Exercise 4. Using the Completeness Theorem “if $\mathcal{T} \models \mathcal{Q}$, then $\mathcal{T} \vdash \mathcal{Q}$ ”, prove that:
“if $\mathcal{T}$ is unsatisfiable, then $\mathcal{T}$ is inconsistent”.

Then prove the converse.

Exercise 5. Recall Theorems A and B:

- Theorem A.

$$\mathcal{T} \models \mathcal{Q} \quad \text{iff} \quad \mathcal{T} \vdash \mathcal{Q}.$$

- Theorem B.

$$\mathcal{T} \text{ is consistent} \quad \text{iff} \quad \mathcal{T} \text{ has a model.}$$

Assume Theorem B. Deduce Theorem A.

Application to finiteness

Exercise 6. Let $\mathcal{L}$ be a language (containing the equality symbol $=$). For every integer $n \geq 1$, let $\mathcal{P}_n$ denote the formula expressing “there are at least n distinct elements”.

1. Write the formulas $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ explicitly.
2. Let $\mathcal{T}$ be a theory with finite models of arbitrarily large size; that is, for every integer k , $\mathcal{T}$ has a model with at least k elements.
Let $\Gamma = \mathcal{T} \cup \{\mathcal{P}_n \mid n \geq 1\}$. Show that every finite subset of Γ has a model.
3. Using the Compactness Theorem, deduce that $\mathcal{T}$ has an infinite model.
4. Conclusion: is there a first-order theory $\mathcal{T}_{\text{finite}}$ whose models are **exactly** all finite structures?

Closure theorem

Exercise 7. For a formula $\mathcal{P}$ whose free variables are denoted by $x_1, \dots, x_n$, its *existential closure* is the formula $\mathcal{P}_{\exists}$ (also denoted by $\exists \mathcal{P}$):

$$\mathcal{P}_{\exists} : \quad \exists x_1 \dots \exists x_n \mathcal{P}$$

1. Show that $\mathcal{P}$ is satisfiable if and only if $\mathcal{P}_{\exists}$ has a model.
2. Prove the equivalence $\neg(\overline{\mathcal{P}}) \equiv (\neg \mathcal{P})_{\exists}$.
Recall that $\overline{\mathcal{P}}$ denotes the universal closure and that $\mathcal{P} \equiv \mathcal{Q}$ means $(\mathcal{P} \models \mathcal{Q} \text{ and } \mathcal{Q} \models \mathcal{P})$; see the chapter “Satisfaction”.
3. Show that if $\Gamma \vdash \mathcal{Q}$, then $\Gamma \vdash \mathcal{Q}_{\exists}$. Is the converse true?