

Compactness Theorem

The Compactness Theorem states that when there are infinitely many assumptions, it is sufficient to use finitely many of them.

This chapter is more difficult than the preceding ones and may be skipped on a first reading.

1. Formulas, satisfiable formulas, logical consequence

1.1. Formulas: definition from above

In the chapter “True/False Logic”, we defined a formula as a finite word constructed from the following elements:

- propositional variables $\text{Var} = \{P_i\}_{i \in I}$,
- parentheses (and),
- and the logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

More precisely:

Definition.

The set Form of **formulas** is constructed according to the following rules:

1. Propositional variables are formulas.
2. If $\mathcal{P}$ and $\mathcal{Q}$ are formulas, then

$$\neg \mathcal{P}, \quad (\mathcal{P} \wedge \mathcal{Q}), \quad (\mathcal{P} \vee \mathcal{Q}), \quad (\mathcal{P} \rightarrow \mathcal{Q}), \quad (\mathcal{P} \leftrightarrow \mathcal{Q})$$

are also formulas.

3. Nothing else is a formula.

This is called the “definition from above”, or the “inductive definition”, of formulas. We saw in the chapter “Which Connectives Are Useful?” that the connectives $\{\neg, \wedge\}$ alone are sufficient to describe every formula. In this chapter, we are interested in the case where the set $\text{Var} = \{P_i\}_{i \in I}$ of propositional variables is infinite but countable. This means that we may take $I = \mathbb{N}$ as the index set for these variables: $P_0, P_1, P_2, \dots$

Remark.

Since there are infinitely many variables, infinitely many indices i would be needed to describe all the formulas. But we want a formula to be a word formed from a finite alphabet. To achieve this, we will use only two symbols to describe all the variables: P and $'$ (prime). Thus we write P for P_0 , P' for P_1 , P'' for P_2 , and so on. A formula is therefore a finite word composed from the finite alphabet $\{P, ', (,), \neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$. Of course, in practice we will retain our notation P_i for clarity.

1.2. Formulas: definition from below

The preceding definition is actually rather abstract and does not describe the set of formulas very well. Here is a new definition, called the “definition from below” or the “recursive definition”.

Definition.

We define sets of formulas Form_n recursively.

- $\text{Form}_0 = \{P_i\}_{i \in I}$ is the set of variables.
- For $n \geq 0$,

$$\text{Form}_{n+1} = \text{Form}_n \cup \{\neg \mathcal{P} \mid \mathcal{P} \in \text{Form}_n\} \cup \{(\mathcal{P} \wedge \mathcal{Q}) \mid \mathcal{P}, \mathcal{Q} \in \text{Form}_n\} \cup \dots$$

(Taking the corresponding union for all the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.)

- $\text{Form} = \bigcup_{n \geq 0} \text{Form}_n$ is the union of all the Form_n .
- The **height** of a formula $\mathcal{P}$ is the least integer n such that $\mathcal{P} \in \text{Form}_n$.

Proposition 1.

If the set $\text{Var} = \{P_i\}_{i \in I}$ of propositional variables is countable (or finite), then the set Form of formulas is also countable.

Proof. The set $\text{Form} = \bigcup_{n \geq 0} \text{Form}_n$ is a union indexed by a countable set, in which each set Form_n is countable, so Form is countable. $\square$

Throughout the rest of this chapter, we will assume that the set of propositional variables is finite or countable, and hence that the set of all formulas is countable.

1.3. Evaluating a formula

We have already seen that evaluating a formula $\mathcal{P}$ consists of assigning a value True or False to each propositional variable P_i , in order to assign a T/F value to the formula $\mathcal{P}$ being evaluated.

Here is the rigorous definition of an evaluation based on the definition of formulas “from above”. Here **1** and **0** denote the two Boolean values T/F.

Definition.

- Let $v : \{P_i\}_{i \in I} \rightarrow \{0, 1\}$ be a function, called an **assignment of truth values** or a **valuation**: for every $i \in I$, we have $v(P_i) = 0$ or $v(P_i) = 1$.
- We extend this function to a function $v : \text{Form} \rightarrow \{0, 1\}$ on the set Form of formulas:
 - $v(\neg \mathcal{P}) = 1$ iff $v(\mathcal{P}) = 0$,
 - $v(\mathcal{P} \wedge \mathcal{Q}) = 1$ iff $v(\mathcal{P}) = 1$ and $v(\mathcal{Q}) = 1$,
 - and according to the same principle for $\vee, \rightarrow, \leftrightarrow$.
- The value $v(\mathcal{P})$ is the **evaluation** of $\mathcal{P}$ under v .

1.4. Satisfiability, logical consequence

Definition.

- A formula $\mathcal{P}$ is **satisfiable** if there exists a valuation v such that $v(\mathcal{P})$ is true. We then say that v **satisfies** $\mathcal{P}$, or that v is a **model** for $\mathcal{P}$.
- A set of formulas $\Gamma = \{\mathcal{P}_j\}_{j \in J}$ is **satisfiable** if there exists a valuation v such that, for every $j \in J$, $v(\mathcal{P}_j)$ is true.
- A formula, or a set of formulas, that is not satisfiable is said to be **unsatisfiable** or **contradictory**.

Definition.

Let Γ be a set of formulas and $\mathcal{Q}$ a formula. We say that $\mathcal{Q}$ is a *logical consequence* of Γ if every valuation v that satisfies Γ also satisfies $\mathcal{Q}$. We write:

$$\Gamma \models \mathcal{Q}$$

This means that $\Gamma \cup \{\neg \mathcal{Q}\}$ is not satisfiable; that is, no assignment of truth values can satisfy both Γ and $\neg \mathcal{Q}$.

We should think of Γ as assumptions and $\mathcal{Q}$ as a conclusion. Whenever all the assumptions hold, the conclusion does as well.

2. Compactness Theorem

2.1. Statement

Let Γ be a finite or infinite set of formulas.

Theorem 1.

Γ is satisfiable if and only if, for every finite subset $\Gamma_0 \subset \Gamma$, Γ_0 is satisfiable.

To make the proof easier to write, we will say that Γ is *finitely satisfiable* if, for every finite subset $\Gamma_0 \subset \Gamma$, Γ_0 is satisfiable. Thus the Compactness Theorem states:

$$\Gamma \text{ satisfiable} \quad \text{iff} \quad \Gamma \text{ finitely satisfiable}$$

Remarks:

1. The forward implication is obvious. If Γ is satisfiable, then every subset of Γ is also satisfiable, so in particular Γ is finitely satisfiable.
2. If Γ is a finite set, then the theorem is obvious; there is nothing to prove.
3. If Γ is infinite, the converse is far from obvious. Some results that hold for finite sets do not necessarily hold for infinite sets. For example, a finite intersection of open intervals is an open interval (possibly empty). However, this is no longer true for an infinite intersection:

$$\bigcap_{n \geq 1}]-\frac{1}{n}, 1 + \frac{1}{n}[= [0, 1]$$

This is a striking example in which an infinite intersection of open intervals results in a closed interval.

2.2. Corollaries

The following corollaries are also called the “Compactness Theorem”.

Corollary 1.

Γ is unsatisfiable if and only if there exists a finite set $\Gamma_0 \subset \Gamma$ such that Γ_0 is unsatisfiable.

In other words, if a set of formulas cannot be satisfied, this cannot be due to the fact that it contains infinitely many formulas.

Here is the positive counterpart:

Corollary 2.

$\Gamma \models \mathcal{Q}$ if and only if there exists a finite set $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \models \mathcal{Q}$.

This is the version that interests us most: if we have infinitely many formulas Γ as assumptions, with the formula $\mathcal{Q}$ as a logical consequence, then we can extract a finite subset Γ_0 that still entails the same logical consequence $\mathcal{Q}$. Thus we go from infinitely many assumptions to finitely many.

2.3. Proof of the corollaries

Without waiting for the proof of the theorem, here is a proof of the corollaries.

Proof of Corollary 1. The reverse implication $\Leftarrow$ is clear: if Γ_0 is not satisfiable, then no set Γ containing Γ_0 will be satisfiable.

The forward implication $\Rightarrow$ is proved using the Compactness Theorem, which states:

$$\forall \Gamma_0 \subset \Gamma \quad \Gamma_0 \text{ satisfiable} \quad \rightarrow \quad \Gamma \text{ satisfiable}$$

(Only finite subsets Γ_0 are considered.) Its contrapositive is therefore:

$$\Gamma \text{ not satisfiable} \quad \rightarrow \quad \text{not } (\forall \Gamma_0 \subset \Gamma \quad \Gamma_0 \text{ satisfiable})$$

This can be written:

$$\Gamma \text{ unsatisfiable} \quad \rightarrow \quad \exists \Gamma_0 \subset \Gamma \quad \Gamma_0 \text{ unsatisfiable}$$

□

Proof of Corollary 2. We will use the fact that “ $\Gamma \models Q$ if and only if $\Gamma \cup \{\neg Q\}$ is unsatisfiable”.

The reverse implication $\Leftarrow$ is the easy one: let Γ_0 be finite with $\Gamma_0 \models Q$. Then $\Gamma_0 \cup \{\neg Q\}$ is unsatisfiable.

For every set Γ containing Γ_0 , it follows a fortiori that $\Gamma \cup \{\neg Q\}$ is unsatisfiable. Thus $\Gamma \models Q$.

Let us prove the forward implication $\Rightarrow$ using Corollary 1. Suppose $\Gamma \models Q$. Then $\Gamma \cup \{\neg Q\}$ is unsatisfiable.

By Corollary 1, there exists a finite set $\Delta_0 \subset \Gamma \cup \{\neg Q\}$ such that Δ_0 is unsatisfiable.

- If $\neg Q \in \Delta_0$, then we write $\Delta_0 = \Gamma_0 \cup \{\neg Q\}$ with $\Gamma_0 \subset \Gamma$ finite. Thus $\Gamma_0 \cup \{\neg Q\}$ is unsatisfiable, which is equivalent to $\Gamma_0 \models Q$.
- If $\neg Q \notin \Delta_0$, then we simply set $\Gamma_0 = \Delta_0$ which is unsatisfiable. A fortiori, $\Gamma_0 \cup \{\neg Q\}$ is unsatisfiable, so $\Gamma_0 \models Q$.

In both cases, we have proved $\Gamma_0 \models Q$.

□

3. Proof

The forward implication $\Rightarrow$ of the Compactness Theorem is clear: indeed, if Γ is satisfiable, then every subset $\Gamma_0 \subset \Gamma$ (finite or not) is satisfiable. We therefore focus on proving the converse: if Γ is finitely satisfiable, then Γ is satisfiable.

3.1. Part 1

Let Γ be a finitely satisfiable set of formulas. First we will construct a set Δ such that:

1. $\Gamma \subset \Delta$,
2. for every formula Q in the set Form of all formulas, either $Q \in \Delta$ or $\neg Q \in \Delta$,
3. Δ is finitely satisfiable.

We will recursively construct increasingly large finitely satisfiable sets Δ_n . The set Δ will be the union of all the Δ_n . Let us begin by defining:

$$\Delta_0 = \Gamma$$

By assumption, Δ_0 is finitely satisfiable.

We assumed that the set Var of propositional variables was countable, so by Proposition 1, the set Form of all formulas is also countable. We can therefore enumerate it using integers:

$$\text{Form} = \{Q_1, Q_2, \dots\}$$

Let us first show how to construct Δ_1 . We distinguish two cases:

- If $\Delta_0 \cup \{Q_1\}$ is finitely satisfiable, we simply set:

$$\Delta_1 = \Delta_0 \cup \{Q_1\}$$

- Otherwise, we set:

$$\Delta_1 = \Delta_0 \cup \{\neg Q_1\}$$

But in this case too, Δ_1 is finitely satisfiable, by the lemma below.

Thus in every case, Δ_1 is finitely satisfiable.

Lemma 1.

Let Σ be a finitely satisfiable set and let Q be any formula. Then $\Sigma \cup \{Q\}$ or $\Sigma \cup \{\neg Q\}$ is finitely satisfiable.

Proof. Suppose for contradiction that $\Sigma \cup \{Q\}$ is not finitely satisfiable and that $\Sigma \cup \{\neg Q\}$ is not finitely satisfiable either. Thus:

- there exists a finite set $\Sigma_1 \cup \{Q\}$ that is unsatisfiable (with $\Sigma_1 \subset \Sigma$),
- and there exists a finite set $\Sigma_2 \cup \{\neg Q\}$ that is unsatisfiable (with $\Sigma_2 \subset \Sigma$).

Let $\Sigma_0 = \Sigma_1 \cup \Sigma_2$. It is a finite set contained in Σ . By assumption, it is therefore satisfiable. Let v be a valuation that satisfies Σ_0 . If v satisfies Q , then $\Sigma_1 \cup \{Q\}$ is satisfied, which is impossible. But if v does not satisfy Q , then v satisfies $\neg Q$ and hence v satisfies $\Sigma_2 \cup \{\neg Q\}$, which is also impossible. We have therefore obtained a contradiction. $\square$

Let us continue our recursive construction. Suppose that we have already constructed a finitely satisfiable Δ_n of the form:

$$\Delta_n = \Gamma \cup \{\varepsilon_1 Q_1, \dots, \varepsilon_n Q_n\}$$

where $\varepsilon_i Q_i$ denotes either Q_i or $\neg Q_i$. We then define:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{Q_{n+1}\} & \text{if it is finitely satisfiable,} \\ \Delta_n \cup \{\neg Q_{n+1}\} & \text{otherwise.} \end{cases}$$

By Lemma 1, Δ_{n+1} is finitely satisfiable.

Now define:

$$\Delta = \bigcup_{n \geq 0} \Delta_n$$

Since $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \dots$, we should think of Δ as the limit of the Δ_n as n tends to $+\infty$. Let us now verify the stated properties:

1. $\Gamma = \Delta_0 \subset \Delta$,
2. for every formula Q_n , either $Q_n \in \Delta_n \subset \Delta$ or $\neg Q_n \in \Delta_n \subset \Delta$,
3. finally, Δ is finitely satisfiable. Indeed, let $\Gamma_0 \subset \Delta$ be a finite set. Then there exists $n \geq 0$ such that $\Gamma_0 \subset \Delta_n$. But then Γ_0 is satisfiable because Δ_n is finitely satisfiable.

3.2. Part 2

We will now prove that all the formulas $\mathcal{P}$ in Δ are simultaneously satisfiable. We will even specify the valuation v that satisfies them. Since $\Gamma \subset \Delta$, it will follow that Γ is satisfiable.

Let us define this valuation v . Write $\text{Var} = \{P_i\}_{i \in I}$ for the set of propositional variables. Let v be the valuation defined by:

$$\begin{cases} v(P_i) = 1 & \text{if } P_i \in \Delta, \\ v(P_i) = 0 & \text{otherwise.} \end{cases}$$

Once defined on the propositional variables, v extends to all formulas (see Section 1.3).

Let $\mathcal{P}$ be any formula. We will prove:

Lemma 2.

$\mathcal{P}$ is satisfied by v if and only if $\mathcal{P} \in \Delta$.

Thus, since $\Gamma \subset \Delta$, every formula $\mathcal{P} \in \Gamma$ is satisfied by v , and so Γ is satisfiable because it is satisfied by v . It remains to prove the lemma.

Proof. The proof is by induction on the height of $\mathcal{P}$. To simplify the proof, suppose that the only connectives used are $\neg$ and $\wedge$. This is possible without loss of generality by the chapter “Which Connectives Are Useful?”

- If $\mathcal{P}$ is one of the propositional variables P_i , then by definition of v , v satisfies $\mathcal{P}$ if and only if $\mathcal{P} \in \Delta$.
- Suppose that the assertion of the lemma holds for the formula $\mathcal{P}$; let us prove that it also holds for $\neg\mathcal{P}$.

$$\begin{aligned}
 (\neg\mathcal{P}) \text{ satisfied by } v &\iff v(\neg\mathcal{P}) = 1 \\
 &\iff v(\mathcal{P}) = 0 \\
 &\iff \text{not}(\mathcal{P} \text{ satisfied by } v) \\
 &\iff \text{not}(\mathcal{P} \in \Delta) \\
 &\iff \mathcal{P} \notin \Delta \\
 &\iff \neg\mathcal{P} \in \Delta
 \end{aligned}$$

For the last line, recall that property (2) of Δ tells us that either $\mathcal{P}$ or $\neg\mathcal{P}$ belongs to Δ .

- Suppose that the assertion of the lemma holds for the formulas $\mathcal{P}$ and $\mathcal{Q}$; let us prove that it also holds for $\mathcal{P} \wedge \mathcal{Q}$.

Forward implication $\Rightarrow$. If $\mathcal{P} \wedge \mathcal{Q}$ is satisfied by v , let us show that $\mathcal{P} \wedge \mathcal{Q} \in \Delta$.

Since v satisfies $\mathcal{P} \wedge \mathcal{Q}$, v satisfies $\mathcal{P}$, so $\mathcal{P} \in \Delta$, and v satisfies $\mathcal{Q}$, so $\mathcal{Q} \in \Delta$.

Suppose for contradiction that $\mathcal{P} \wedge \mathcal{Q} \notin \Delta$. Then, by property (2) of Δ , we have $\neg(\mathcal{P} \wedge \mathcal{Q}) \in \Delta$. Now consider the set $\{\mathcal{P}, \mathcal{Q}, \neg(\mathcal{P} \wedge \mathcal{Q})\}$. This is a finite, unsatisfiable set of formulas (no valuation can simultaneously satisfy $\mathcal{P}$, $\mathcal{Q}$, and $\neg(\mathcal{P} \wedge \mathcal{Q})$). This contradicts the fact that Δ is finitely satisfiable.

Conclusion: $\mathcal{P} \wedge \mathcal{Q} \in \Delta$.

Reverse implication $\Leftarrow$. If $\mathcal{P} \wedge \mathcal{Q} \in \Delta$, let us show that $\mathcal{P} \wedge \mathcal{Q}$ is satisfied by v .

Suppose for contradiction that $\mathcal{P} \notin \Delta$. Then $\neg\mathcal{P} \in \Delta$ (again by property (2) of Δ). Thus $\{\mathcal{P} \wedge \mathcal{Q}, \neg\mathcal{P}\} \subset \Delta$ is a finite set, but of course it is not satisfiable. This again contradicts the fact that Δ is finitely satisfiable. Thus $\mathcal{P} \in \Delta$, so by the induction hypothesis $v(\mathcal{P}) = 1$.

Similarly, we prove that $\mathcal{Q} \in \Delta$ and hence $v(\mathcal{Q}) = 1$. But then $v(\mathcal{P} \wedge \mathcal{Q}) = 1$ (by the construction of the valuation), and so $\mathcal{P} \wedge \mathcal{Q}$ is satisfied by v .

□

Remark. If we allow the other classical connectives $\vee$, $\rightarrow$, $\leftrightarrow$, then we can prove in the same way that $\mathcal{P} \vee \mathcal{Q}$ is satisfied by v if and only if $\mathcal{P} \vee \mathcal{Q} \in \Delta$, and so on. (This is left as an exercise.)

4. $\{0, 1\}^{\mathbb{N}}$ is compact

Why is the Compactness Theorem called by this name? Here we make the connection with the notion of a compact set. This section is intended for readers who already know topology, whose concepts will not be reviewed here.

4.1. Topology of $\{0, 1\}^{\mathbb{N}}$

- Let $\mathbb{B} = \{0, 1\}$ have the discrete topology; that is, $\{0\}$ and $\{1\}$ are both open and closed sets.
- $\mathbb{B}^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ is a product space. It may be viewed as the set $\{v : \mathbb{N} \rightarrow \{0, 1\}\}$ of functions that assign a value 0 or 1 to each integer i . In other words, it is the set of valuations that assign a T/F value to each propositional variable P_i (here $\text{Var} = \{P_i\}_{i \in \mathbb{N}}$ is countable).
- We equip $\{0, 1\}^{\mathbb{N}}$ with the product topology; by definition, this is the coarsest topology that makes the projections $\pi_i : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}$ continuous.
- Thus, for fixed i , $\pi_i^{-1}(\{1\}) = \{v : \mathbb{N} \rightarrow \{0, 1\} \mid v(i) = 1\}$ is closed because it is the inverse image of a closed set under a continuous map. It is also open because it is the inverse image of an open set, which makes its complement closed.
- Let $\mathcal{P}$ be a formula. Write $V(\mathcal{P}) = \{v \in \{0, 1\}^{\mathbb{N}} \mid v(\mathcal{P}) = 1\}$. The set $V(\mathcal{P})$ is closed. The proof is by induction on the height of $\mathcal{P}$ (much as in the second part of the proof of the Compactness Theorem; see the exercises).
- A formula $\mathcal{P}$ is satisfiable if and only if $V(\mathcal{P})$ is nonempty.

4.2. A product of compact spaces is compact

Theorem 2 (Tychonoff's theorem).

An arbitrary product of compact spaces is compact.

The product may be finite or infinite (countable or not).

Corollary 3.

$\{0, 1\}^{\mathbb{N}}$ is a compact space.

Indeed, $\{0, 1\}$ is compact because it is a finite set.

Here is an important property of compact spaces.

Proposition 2.

Let F_j , $j \in J$, be closed subsets of a compact space E . If $\bigcap_{j \in J} F_j = \emptyset$, then there exists a finite set $J_0 \subset J$ such that $\bigcap_{j \in J_0} F_j = \emptyset$.

This is equivalent to the more familiar version for open sets: “Every open cover of a compact space has a finite subcover.”

4.3. Proof of the Compactness Theorem

Let $\Gamma = \{\mathcal{P}_0, \mathcal{P}_1, \dots\}$ be a finitely satisfiable set of formulas.

Write $F_j = V(\mathcal{P}_j)$, which is a closed subset of $\{0, 1\}^{\mathbb{N}}$. Note that F_j is nonempty because $\mathcal{P}_j$ is satisfiable (since $\{\mathcal{P}_j\}$ is a finite subset of Γ). More generally, if $\Gamma_0 = \{\mathcal{P}_j\}_{j \in J_0}$ is a finite set of formulas contained in Γ , then there exists a valuation v that simultaneously satisfies all the $\mathcal{P}_j$. Therefore $v \in F_j$ for every $j \in J_0$. In other words:

$$\bigcap_{j \in J_0} F_j \neq \emptyset$$

Thus every finite intersection of closed sets is nonempty. By the contrapositive of Proposition 2, we therefore have:

$$\bigcap_{j \in \mathbb{N}} F_j \neq \emptyset$$

This means that there exists $v : \mathbb{N} \rightarrow \{0, 1\}$ such that $v(P_j) = 1$ for every $j \in \mathbb{N}$. Thus v satisfies Γ . We have indeed proved that Γ is satisfiable.

Exercises

Formulas, satisfiable formulas, logical consequence

Exercise 1. Consider $\text{Var} = \{P, Q, R\}$ and $\text{Conn} = \{\neg, \wedge\}$. List all formulas of height 0, then of height 1. How many formulas of height 2 are there? Answer the same questions with $\text{Var} = \{P, Q\}$ and $\text{Conn} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$.

Exercise 2. Show that two formulas are logically equivalent exactly when each is a logical consequence of the other.

$$\mathcal{P} \equiv \mathcal{Q} \quad \text{iff} \quad \mathcal{P} \models \mathcal{Q} \text{ and } \mathcal{Q} \models \mathcal{P}$$

Exercise 3. Find all valuations defined on $\{P_i\}_{i \in \mathbb{N}}$ that satisfy the infinite set of formulas:

$$\Gamma = \{P_0 \rightarrow P_1, P_1 \rightarrow P_2, P_2 \rightarrow P_3, \dots\}$$

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Exercise 4. For the variables $\{P_i\}_{i \in \mathbb{N}}$, define, for $n \geq 0$, the formula $\mathcal{P}_n = \neg(P_n \wedge P_{n+1})$. Let $\Gamma = \{\mathcal{P}_n \mid n \geq 0\}$.

1. Show that every finite subset of Γ is satisfiable.
2. Deduce that Γ is satisfiable.
3. Characterize all valuations defined on $\{P_i\}_{i \in \mathbb{N}}$ that satisfy Γ .

Exercise 5. For each integer $n \in \mathbb{N}$, define the formula:

$$\mathcal{P}_n = (P_n \vee P_{n+1}) \wedge (\neg P_{n+2}).$$

Consider the set $\Gamma = \{\mathcal{P}_n \mid n \geq 0\}$.

1. For fixed $n \geq 0$, show that $\{\mathcal{P}_n\}$ is satisfiable.
2. For fixed $n \geq 0$, show that $\{\mathcal{P}_n, \mathcal{P}_{n+1}\}$ is satisfiable.
3. Is Γ satisfiable?

Exercise 6. Starting from Corollary 2, which may be assumed, give a proof of the Compactness Theorem.

Proof

Exercise 7. Extend the proof of Lemma 2 to show directly that:

$$v \text{ satisfies } \mathcal{P} \vee \mathcal{Q} \quad \text{iff} \quad \mathcal{P} \vee \mathcal{Q} \in \Delta$$

$\{0, 1\}^{\mathbb{N}}$ is compact

Exercise 8. Let $\mathcal{P}$ be a formula and $V(\mathcal{P}) = \{v \in \{0, 1\}^{\mathbb{N}} \mid v(\mathcal{P}) = 1\}$. Show that $V(\mathcal{P})$ is a closed subset of $\{0, 1\}^{\mathbb{N}}$. First justify this for propositional variables, then prove it by induction on the height of $\mathcal{P}$, using the proof of Lemma 2 as a guide.